Wednesday, April 28, 2021

Magnetic vector potential (summary)

definition: $$\boxed{\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})}$$

Coulomb gauge: $$\boxed{\vec{\nabla} \cdot \vec{A} = 0}$$
(typically used in magnetostatics)

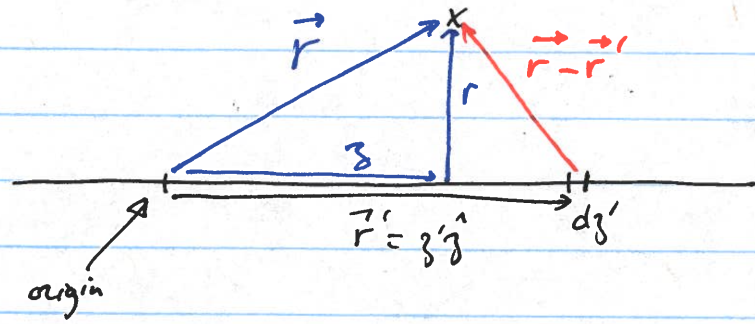
Source equation: $$\boxed{\nabla^2 \vec{A}(\vec{r}) = -\mu_0 \vec{J}(\vec{r})}$$
(in Coulomb gauge)

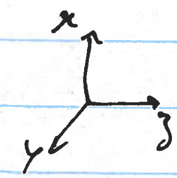
Integral source equation: $$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,')\, d^3r'}{|\vec{r} - \vec{r}\,'|}}$$
(in Coulomb gauge)

Example: calculate $\vec{A}$ for an infinitely long thin wire with current I.

$$\vec{J}(\vec{r}\,')\, d^3r' = I\, d\vec{\ell}\,' = I\, dz'\, \hat{z}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{dz'}{|\vec{r}-\vec{r}'|}\,\hat{z} \qquad \text{(in Coulomb gauge)}$$





$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{z} \int_{-\infty}^{+\infty} \frac{dz'}{\sqrt{r^2+(z'-z)^2}} = \frac{\mu_0 I}{4\pi} I \hat{z} \int_{-\infty}^{+\infty} \frac{du}{\sqrt{r^2+u^2}}$$

substitution
$u = z'-z$
$du = dz'$

same integral as on midterm problem #1

$$= \frac{\mu_0 I}{4\pi}\hat{z}\, \underbrace{\ln\left[u+\sqrt{r^2+u^2}\right]\Big|_{-\infty}^{+\infty}}_{\text{diverges}}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z} \lim_{L\to+\infty} \ln\left[u+\sqrt{r^2+u^2}\right]\Big|_{-L}^{+L}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z} \lim_{L\to+\infty} \underbrace{\ln\left\{\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right\}}_{\ln\left\{\frac{1+\sqrt{1+(r/L)^2}}{-1+\sqrt{1+(r/L)^2}}\right\}}$$

$$= \ln\Big[1+\underbrace{\sqrt{1+(r/L)^2}}_{1+\frac{1}{2}\left(\frac{r}{L}\right)^2+\cdots}\Big] - \ln\Big[-1+\underbrace{\sqrt{1+(r/L)^2}}_{1+\frac{1}{2}\left(\frac{r}{L}\right)^2+\cdots}\Big]$$

$$\simeq \ln\left[2+\frac{1}{2}\left(\frac{r}{L}\right)^2+\cdots\right] - \ln\left[\frac{1}{2}\left(\frac{r}{L}\right)^2+\cdots\right]$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{z} \lim_{L\to+\infty} \left\{ \underbrace{\ln\left[2+\frac{1}{2}\left(\frac{r}{L}\right)^2 + \cdots\right]}_{\to \ln(2)} - \underbrace{\left(\ln\frac{1}{2} + \ln\left(r/L\right)^2 + \cdots\right)}_{-\ln 2 + 2\ln\left(\frac{r}{L}\right)} \right\}$$

$$= -\ln 2 + 2\ln(r) - 2\ln(L)$$

$$\simeq \frac{\mu_0 I}{4\pi}\hat{z} \lim_{L\to\infty} \underbrace{2\ln 2 + 2\ln L}_{\text{diverging constant (independent of } x,y,z)} - 2\ln r$$

↳ does not affect $\vec{B} = \vec{\nabla}\times\vec{A}$

↳ ignore

$$\Rightarrow \boxed{\vec{A}(\vec{r}) = -\frac{\mu_0 I}{4\pi}\ln(r)\hat{z}}$$

note: Quantization of the EM field is often done with $\vec{A}$

becomes an operator

harmonic oscillator

$a^\dagger$ & a

Multipole expansion of $\vec{A}$

In the Coulomb gauge:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}'}{\underbrace{|\vec{r}-\vec{r}'|}}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

(for $r > r'$)

angle between $\vec{r}$ & $\vec{r}'$

$$= \frac{\mu_0 I}{4\pi} \left[\underbrace{\frac{1}{r} \oint d\vec{l}'}_{\substack{\text{monopole} \\ =0}} + \underbrace{\frac{1}{r^2} \oint r' \cos\alpha \, d\vec{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3} \oint r'^2 \left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\vec{l}'}_{\text{quadrupole}} + \ldots \right]$$

since $\oint d\vec{l}' = 0$

Magnetic dipole

At large distances the magnetic dipole term typically dominates.
↳ the character of most magnetic fields is dipole-like.

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \underbrace{\oint (\vec{r}' \cdot \hat{r}) d\vec{l}'}_{-\hat{r} \times \int_{S \text{ of loop}} d\vec{s}'} = \frac{\mu_0}{4\pi} \frac{1}{r^2} \left(I \int_{S \text{ of loop}} d\vec{s}' \right) \times \hat{r}$$

see problem 1.62
problem set #4

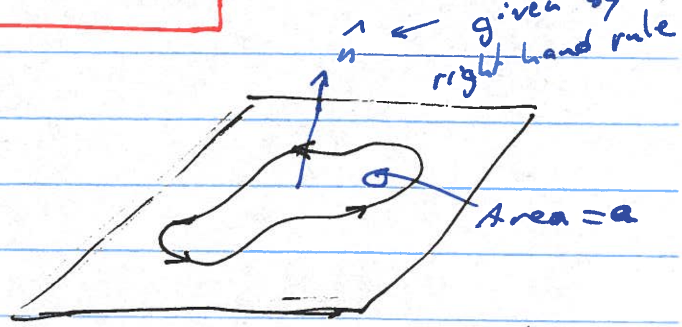
$$= \left(\int_{S \text{ of loop}} d\vec{s}' \right) \times \hat{r}$$

if we define the <u>magnetic moment</u> as

$$\vec{m} = I \underbrace{\int_{S \text{ of loop}} d\vec{s}\,'}_{\text{vector area} = \vec{a}} = \boxed{I\,\vec{a} = \vec{m}}$$

then
$$\boxed{\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}}$$

for a planar current



the vector area is $\vec{a} = \text{area}\ \hat{n}$

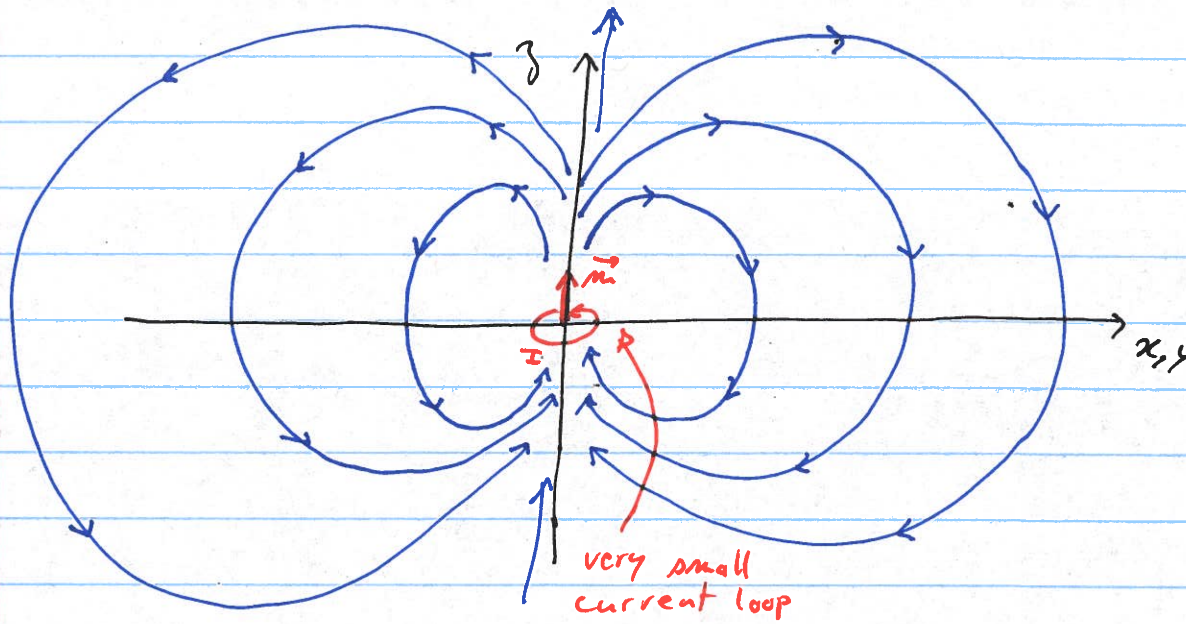
<u>Alternate forms:</u> $\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2}\,\hat{\phi}$

$$\vec{B}_{dipole}(\vec{r}) = \vec{\nabla} \times \vec{A}_{dipole} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

for $\vec{m} = m\hat{z}$

$$\Rightarrow \boxed{\vec{B}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]}$$

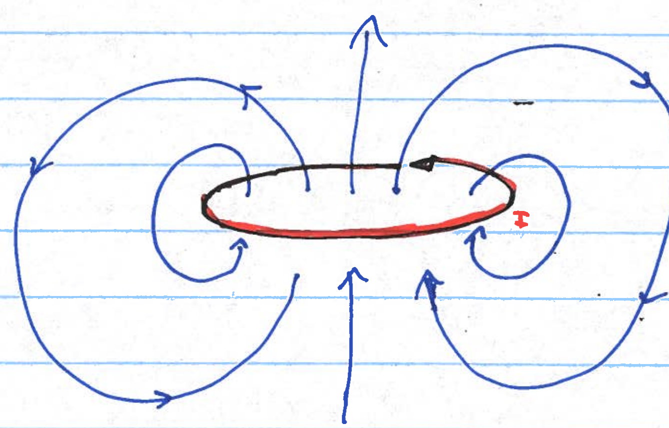
Compare with $\vec{E}_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] \Rightarrow$ same form magnetic field of an ideal magnetic dipole:



Important magnet coil structures

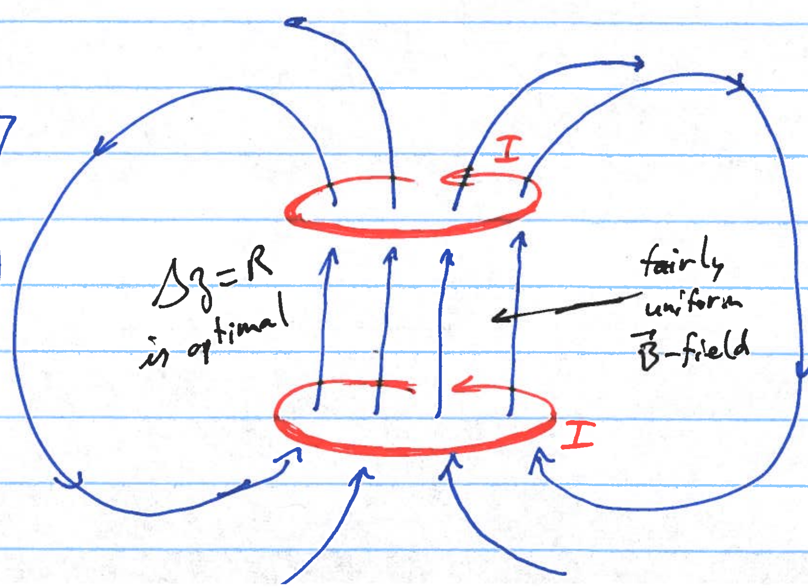
Single coil:

[similar to an ideal magnetic dipole]



Helmholtz coil pair:

[far away, the B-field is close to an idea magnetic dipole]

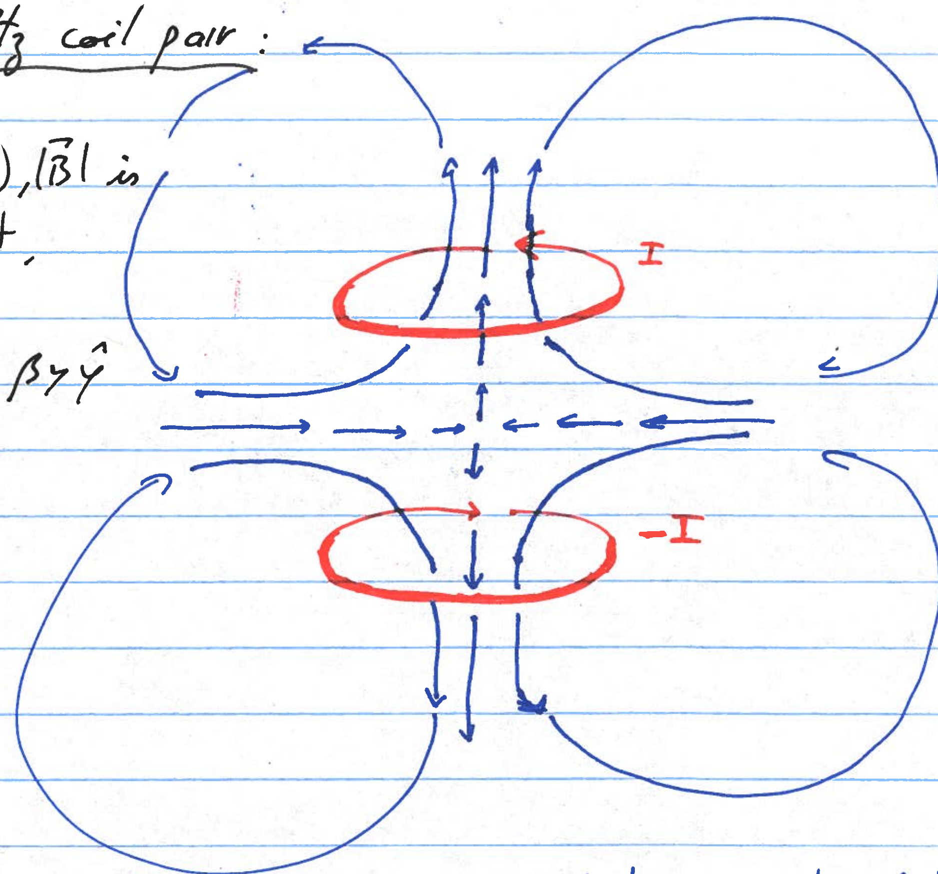


Anti-Helmholtz coil pair:

near origin (center), $|\vec{B}|$ is linear. In fact,

$$\vec{B} = -\beta x \hat{x} - \beta y \hat{y} + 2\beta z \hat{z}$$

with $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$



quadrapole magnetic field

note: An anti-H coil produces a $|\vec{B}|$ minimum, and it can be used to trap "weak field seeking $\underbrace{\text{spin states}}_{\text{magnetic moments}}$"