Wednesday, May 5, 2021

Matching conditions for $\vec{B}$ across a surface current

Electrostatics

$$\vec{E}_{\substack{\text{surface}\\\text{of charge}}} = \frac{\sigma_0}{2\varepsilon_0}\,\hat{n} \qquad \left(\begin{array}{c}\text{no external}\\ E\text{-fields}\end{array}\right)$$

External E-fields can be present

$$\Delta\vec{E}\Big|_{\text{surface}} = \vec{E}_1\Big|_s - \vec{E}_2\Big|_s = \frac{\sigma_0}{\varepsilon_0}\,\hat{n}_1$$

$$\vec{E}_{1,\parallel} = \vec{E}_{2,\parallel}$$

$$\Delta\vec{E}\Big|_s \cdot \hat{n}_1 = \frac{\sigma_0}{\varepsilon_0}$$

Qualitatively:

→ Normal/perpendicular component is affected

→ parallel component is unaffected (i.e. is continuous)

V is continuous across boundary

Magnetostatics

$$\vec{B}_{\substack{\text{surface}\\\text{current}}} = \frac{\mu_0}{2}\,\vec{K}\times\hat{n} \qquad \left(\begin{array}{c}\text{no ext.}\\ B\text{-fields}\end{array}\right)$$

$$\left.\begin{array}{l}\Delta\vec{B} = \vec{B}_1 - \vec{B}_2 = \mu_0\,\vec{K}\times\hat{n}_1 \\[2ex] B_{1,\parallel} - B_{2,\parallel} = \mu_0 K\end{array}\right\}\ \text{from Ampère's Law}$$

"$\parallel$" → $\left\{\begin{array}{l}\parallel \text{ to surface}\\ \perp \text{ to } \vec{K}\end{array}\right.$

consistent

$$\Delta B\Big|_s \cdot \hat{n}_1 = 0 \qquad (\text{from } \vec{\nabla}\cdot\vec{B} = 0)$$

→ perpendicular component is unaffected (i.e. is continuous)

→ parallel component to surface is affected (but not the one parallel to $\vec{K}$)

$\vec{A}_\parallel$ is continuous (from Ampère's law) $\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}_{\text{enclosed}}\cdot d\vec{a}$

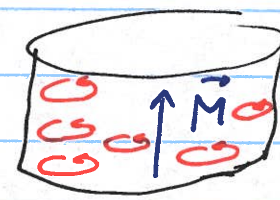
$\vec{A}_\perp$ is continuous in Coulomb gauge.

note: $\frac{\partial A}{\partial n}$ is discontinuous.

Magnetism in matter

Basic model: Matter is made up of small magnetic moments

i.e. $\underbrace{\text{small current loops}}_{\text{in reality electron spins}}$.

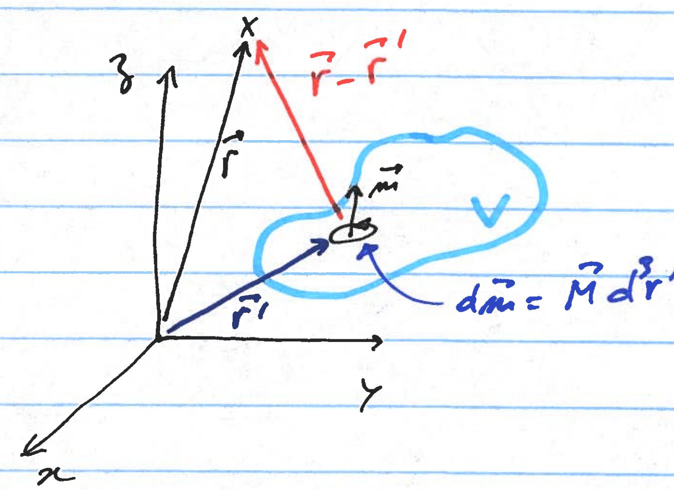


$\underbrace{\text{Magnetization}}_{\text{magnetic equivalent of polarization } \vec P}$ $\vec M$ = total magnetic moment per unit volume

Bound Currents

recall:

$$\vec A_{dipole}(\vec r) = \frac{\mu_0}{4\pi}\,\frac{\vec m \times \widehat{(\vec r-\vec r')}}{|\vec r-\vec r'|^2}$$



$$\vec A_M(\vec r) = \frac{\mu_0}{4\pi}\int_V \frac{\vec M(\vec r') \times \widehat{(\vec r-\vec r')}}{|\vec r-\vec r'|^2}\,d^3r'$$

however, $$\frac{\widehat{(\vec r-\vec r')}}{|\vec r-\vec r'|^2} = \vec\nabla_{r'}\frac{1}{|\vec r-\vec r'|}$$ [see also Wednesday, March 31 lecture]

thus $$\vec M(\vec r') \times \frac{\widehat{(\vec r-\vec r')}}{|\vec r-\vec r'|^2} = \vec M(\vec r') \times \vec\nabla_{r'}\frac{1}{|\vec r-\vec r'|}$$

$$= \frac{1}{|\vec r-\vec r'|}\vec\nabla\times\vec M - \vec\nabla_{r'}\times\left(\frac{\vec M(\vec r')}{|\vec r-\vec r'|}\right)$$

inside front cover of Griffiths

Thus,

$$\vec{A}_M(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{\nabla}\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3r' - \frac{\mu_0}{4\pi}\underbrace{\int_V \vec{\nabla}\times\left(\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)d^3r'}_{-\oint_S \frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\times d\vec{s}\,'}$$

variation on Stoke's theorem see Problem Set #2 problem 4a

$d\vec{s}\,' = \hat{n}'\,ds'$

identify $\vec{J}_M = \vec{J}_b$

$$\Rightarrow \vec{A}_M(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{\nabla}\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3r' + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{M}(\vec{r}\,')\times\hat{n}'}{|\vec{r}-\vec{r}\,'|}\,ds'$$

identify $\vec{K}_M = \vec{K}_b$

recall: $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}$

Magnetization bound volume current: $\boxed{\vec{J}_b = \vec{\nabla}\times\vec{M} = \vec{J}_M}$

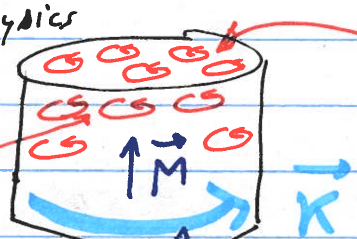
Magnetization bound surface current: $\boxed{\vec{K}_b = \vec{M}\times\hat{n} = \vec{K}_M}$

note 1: $\vec{\nabla}\cdot\vec{J}_b = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{J}_b = 0} \Rightarrow \frac{\partial \rho_b}{\partial t} = 0$

bound charge is conserved

no bound charge leaves material

note 2: Basic Physics



<u>note 3:</u> The surface is where the action is (as in dielectrics)

The Auxiliary Field $\vec{H}$

Total current density: $\vec{J} = \vec{J}_{total} = \vec{J}_b + \underbrace{\vec{J}_{free}}_{\vec{J}_f}$

$\vec{J}_f \leftarrow$ applied by experimentalist (e.g. Ohm's law)

<u>Ampère's law:</u>

$$\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla}\times\vec{M}$$

$$\Rightarrow \quad \vec{\nabla}\times\underbrace{\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right)}_{\vec{H}} = \vec{J}_{free}$$

Auxiliary field $\vec{H}$: $\boxed{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}}$ obeys "Ampère's law"

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_{free}}$$

$$\Rightarrow \oint \vec{H}\cdot d\vec{l} = I_{free,\ enclosed}$$

⚠

$$\vec{\nabla}\cdot\vec{H} = \frac{1}{\mu_0}\cancelto{0}{\vec{\nabla}\cdot\vec{B}} - \vec{\nabla}\cdot M$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \neq 0$$

especially on surface where $\vec{K}_b \neq 0$

Linear Magnetic Media

(local magnetization is proportional to local field)

$$\boxed{\vec{M} = \chi_m \vec{H}}$$ where χ_m = magnetic susceptibility (dimensionless, no units)

not $\vec{B}$

$\chi_m > 0 \Rightarrow$ paramagnetic material

$\chi_m < 0 \Rightarrow$ diamagnetic material

<u>Note</u>: χ_m is generally very small

<u>examples</u>:

<u>diamagnetic</u>	<u>paramagnetic</u>
$\chi_{m_{Cu}} \simeq -10^{-5}$	$\chi_{m_{Al}} \simeq 2\times10^{-5}$
$\chi_{m_{H_2O}} \simeq -9\times10^{-6}$	$\chi_{m_{O_2}}$ (gas) $\sim 10^{-6}$
χ_{m} Pyrolytic carbon $\sim -40\times10^{-5}$	χ_{m} Gadolinium $= 0.48$

Ferromagnetic $\begin{cases} \chi_{m_{Fe}} \sim 10^{5} \\ \chi_{m_{Ni}} \sim 10^{2} \end{cases}$

Thus, $\vec{B} = \mu_0(\vec{H}+\vec{M}) \Rightarrow \boxed{\vec{B} = \underbrace{\mu_0(1+\chi_m)}_{\mu}\vec{H}}$

$\mu = \mu_0(1 + \chi_m)$ = magnetic permeability of materia

$$\Rightarrow \boxed{\vec{B} = \mu \vec{H}}$$

⚠ $\vec{\nabla} \cdot \vec{H}_{linear} = \vec{\nabla} \cdot \left(\frac{1}{\mu}\vec{B}\right) = \frac{1}{\mu} \cancel{\vec{\nabla} \cdot \vec{B}}^{\,0} + \vec{B} \cdot \vec{\nabla}\frac{1}{\mu}$

$\vec{\nabla}\frac{1}{\mu}$ = 0 inside a uniform material

$\vec{\nabla}\frac{1}{\mu}$ ≠ 0 on the surface of a material

$$\Rightarrow \vec{\nabla} \cdot \vec{H}_{linear} \neq 0 \text{ on surfaces}$$

N.B. For most materials (non magnetic) $\mu \simeq \mu_0$

↳ most materials have very little effect magnetic field

↳ most materials are "transparent" to B-field (i.e. you can ignore the materials)

[note: you generally cannot ignore the presence of dielectric.]

Major Exceptions: mu-metal: $\left.\frac{\mu}{\mu_0}\right|_{\mu\text{-metal}} \sim 10^5$ or higher

used for magnetic shielding

Ferromagnetic materials: $\left.\frac{\mu}{\mu_0}\right|_{Ni, Fe, Co} \sim 10^2 - 10^5$