Monday, February 8, 2021

## VI Spherical coordinates (continued)

Volume element: $\boxed{dV = r^2 \sin\theta \, dr \, d\theta \, d\phi}$

replace $dx\,dy\,dz$ with this expression.

⚠ note: $dx\,dy\,dz \neq r^2 \sin\theta \, dr \, d\theta \, d\phi$

more generally: 
$$dV = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}}_{\text{determinant of the Jacobian matrix} \; = r^2 \sin\theta} dr \, d\theta \, d\phi$$

for cylindrical coordinates: $dV = r \, dr \, dz \, d\phi$

## Grad, Div, Curl, etc

Gradient: 
$$\vec{\nabla} f(\vec{r}) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

lots of algebra

$$= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

depends on location!

ex: $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\underbrace{\frac{\partial r}{\partial x}}_{\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\phi} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\phi$$

also ~~$\sin\theta$~~ $\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$

etc...

#3

# SPHERICAL AND CYLINDRICAL COORDINATES

## Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

## Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

**Spherical.** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}};\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient:*
$$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

*Divergence:*
$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

*Curl:*
$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian:*
$$\nabla^2 t = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)}_{\frac{1}{r}\frac{\partial^2}{\partial r^2}(rt)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

**Cylindrical.** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}};\quad d\tau = s\,ds\,d\phi\,dz$

*Gradient:*
$$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

*Divergence:*
$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

*Curl:*
$$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

*Laplacian:*
$$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

## VII Divergence of $\vec{A}=\hat{r}/r^2$ [⚠ plays a central role in E & M]

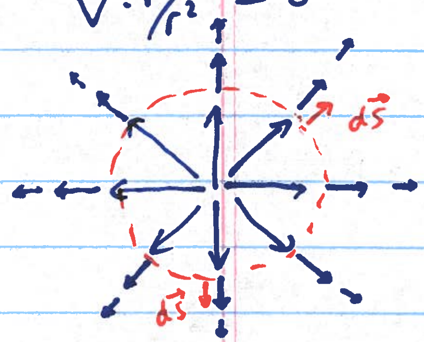
Divergence in spherical coordinates:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right)} + \cancelto{0}{\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\cancelto{0}{\frac{\partial}{\partial r}(1)} = \frac{0}{r^2}$$

$$= 0$$ ⚠ Wrong

$\vec{\nabla}\cdot\hat{r}/r^2 \stackrel{?}{=} 0$




"Paradox":

$$\int_{V=\text{sphere of radius } R}\left(\vec{\nabla}\cdot\vec{A}\right)dV = \int_{S=\text{surface of sphere}}\vec{A}\cdot d\vec{s} = 4\pi R^2\,\frac{1}{r^2}\Big|_{R} = 4\pi$$

$$\int_V \text{"}0\text{"}\,dV = 0 \neq 4\pi$$

does not depend on "R"

or even if $R=\epsilon$ "infinitesimal sphere" centered on origin

Resolution: $$\boxed{\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\,\delta^3(\vec{r})}$$

where $\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$

<u>note</u> : $\vec{\nabla}\cdot\Big(\underbrace{\vec{\nabla}\frac{1}{r}}_{-\frac{\hat{r}}{r^2}}\Big) = \nabla^2\frac{1}{r} = -4\pi\,\delta^3(\vec{r})$

## VIII Helmholtz Theorem

<u>Short version</u> : If you know $\vec{\nabla}\cdot\vec{F}$ and $\vec{\nabla}\times\vec{F}$, as well as the value of $\vec{F}$ on some boundary (e.g. $\vec{F}(r\to+\infty)\to 0$) then $\vec{F}$ is <u>uniquely</u> determined.

<u>Long version</u> :

An arbitrary vector field $\vec{F}(\vec{r})$ can always be decomposed into a sum of 2 vector fields such that

$$\vec{F}(\vec{r}) = \vec{F}_\perp + \vec{F}_{\parallel}$$

with $\vec{\nabla}\cdot\vec{F}_\perp = 0$ and $\vec{\nabla}\times\vec{F}_{\parallel} = \vec{0}$

and more precisely as

$$\boxed{\vec{F}(\vec{r}) = \underbrace{\overset{\text{curl}}{\vec{\nabla}\times\vec{U}(\vec{r})}}_{\vec{F}_\perp} \overset{+}{-} \underbrace{\overset{\text{gradient}}{\vec{\nabla}\,\Omega(\vec{r})}}_{\vec{F}_{\parallel}}}$$

So long as the integrals converge, then $\vec{u}$ and $\Omega$ are "uniquely" given by

$$\Omega(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}_{\vec{r}'} \cdot \vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \text{scalar potential}$$

$$\vec{u}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}_{\vec{r}'} \times \vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \text{vector potential}$$

note: $\vec{F}(\vec{r})$ must fall off faster than $1/r$ for $r \to +\infty$

proof: see PHYS 610 lecture notes (Spring 2018) for March 1, 2018

↳ proof uses $\nabla^2 \frac{1}{r} = -4\pi\,\delta^3(\vec{r})$

Corollary 1: If $\vec{\nabla}\times\vec{F} = 0$, then $\vec{F}(\vec{r})$ can be derived from a scalar potential ⟶ electrostatic potential $V$ for electric field $\vec{E}$.

Corollary 2: If $\vec{\nabla}\cdot\vec{F} = 0$, then $\vec{F}(\vec{r})$ can be derived from a vector potential. ⟶ magnetic vector potential $\vec{A}$ for the magnetic field $\vec{B}$.