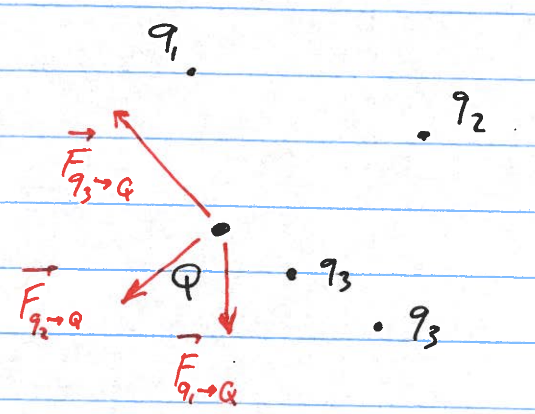
Wednesday, February 10, 2021

Electrostatics [Chapter 2]

Consider a collection of stationary point charges q_1, q_2, q_3, etc.... and a stationary test charge Q.

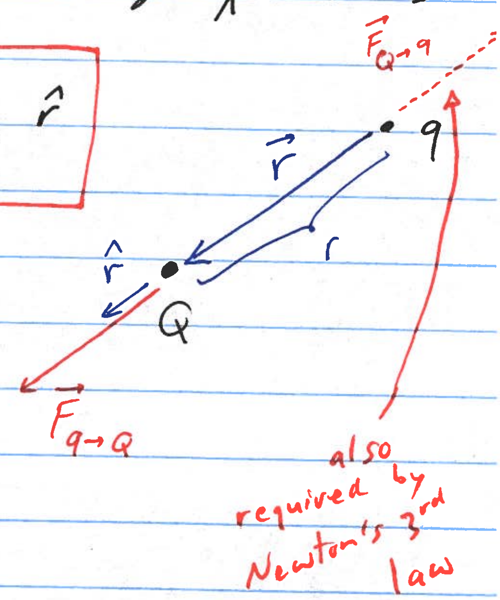


Q1: What is the force on Q due to q_1 (or q_2 or q_3)?

Q2: What is the combined force on Q due to $q_1, q_2,$ and q_3?

A1: For stationary charges, the force is given by Coulomb's law:

$$\boxed{\vec{F}_{q \to Q} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r}}$$



ε_0 = permittivity of free space

$= \underbrace{8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}}_{\text{SI units}}$

Charge (Q, q) is measured in Coulombs [SI unit].

Distance r is measured in meters [SI unit].

<u>ex1:</u> How strong is the Coulomb force between two 1 C charges separated by 1 m. (i.e. $q = 1C$, $Q = 1C$, $r = 1\,m$)

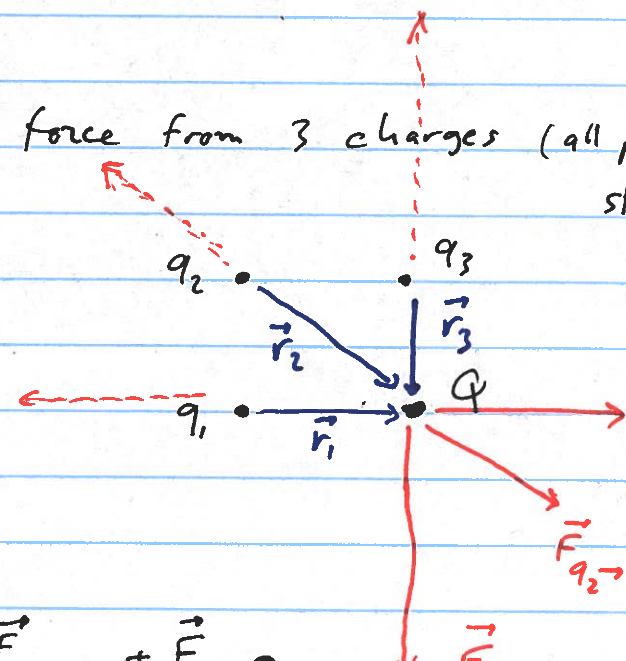
$$|\vec{F}| = \frac{1}{4\pi(8.85\times10^{-12})}\frac{(1)(1)}{(1)^2} = 8.992\times10^{9}\ N$$

$\Rightarrow$ the electric force is very strong (compared to gravity)

<u>Note 1:</u> 1 C is a very large charge.
$-1C = 6.24\times10^{18}$ electrons

<u>Note 2:</u>
- Charges of the same sign repel.
- charges of the opposite sign attract.

<u>ex 2:</u> Coulomb force from 3 charges (all positive & stationary)



$$\vec{F}_{q_1\to Q} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{r_1^2}\hat{r}_1$$

$$\vec{F}_{q_2\to Q} = \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{r_2^2}\hat{r}_2$$

$$\vec{F}_{q_3\to Q} = \frac{1}{4\pi\epsilon_0}\frac{q_3 Q}{r_3^2}\hat{r}_3$$

<u>Total Force:</u>

$$\vec{F}_{total} = \vec{F}_{q_1\to Q} + \vec{F}_{q_2\to Q} + \vec{F}_{q_3\to Q}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \frac{q_3}{r_3^2}\hat{r}_3\right)$$

Generalization: for N charges

$$\vec{F}_{total} = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i \hat{r}_i}{r_i^2} = \vec{E}$$

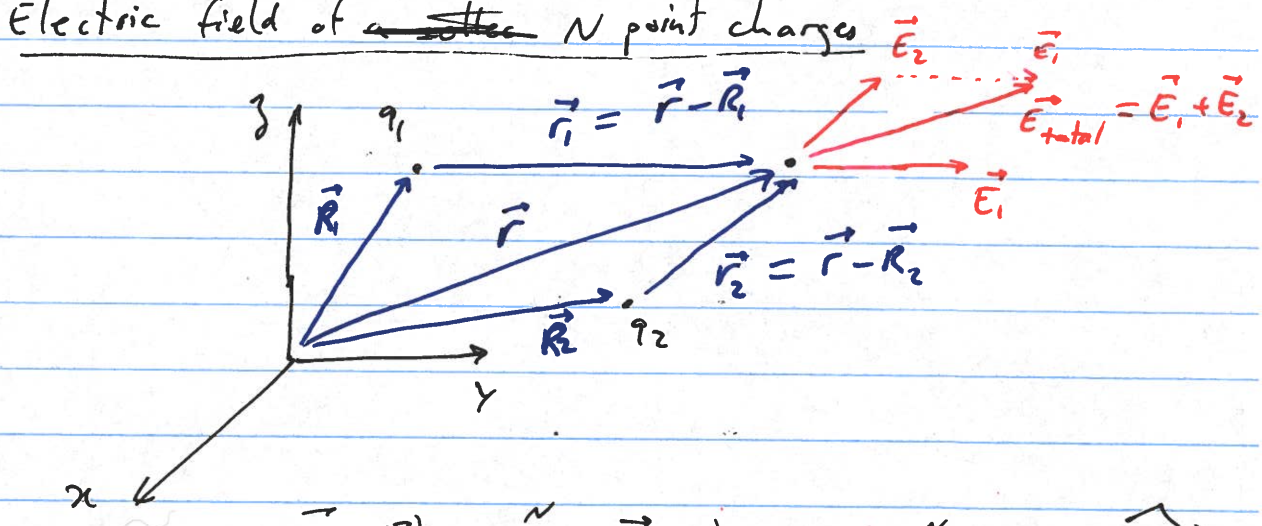
$$= Q\vec{E}$$

Definition: the electric field on a test charge Q is

$$\vec{E} = \frac{\vec{F}_{total}}{Q}$$

Electric field of a point charge (at origin): $\vec{E}_q(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

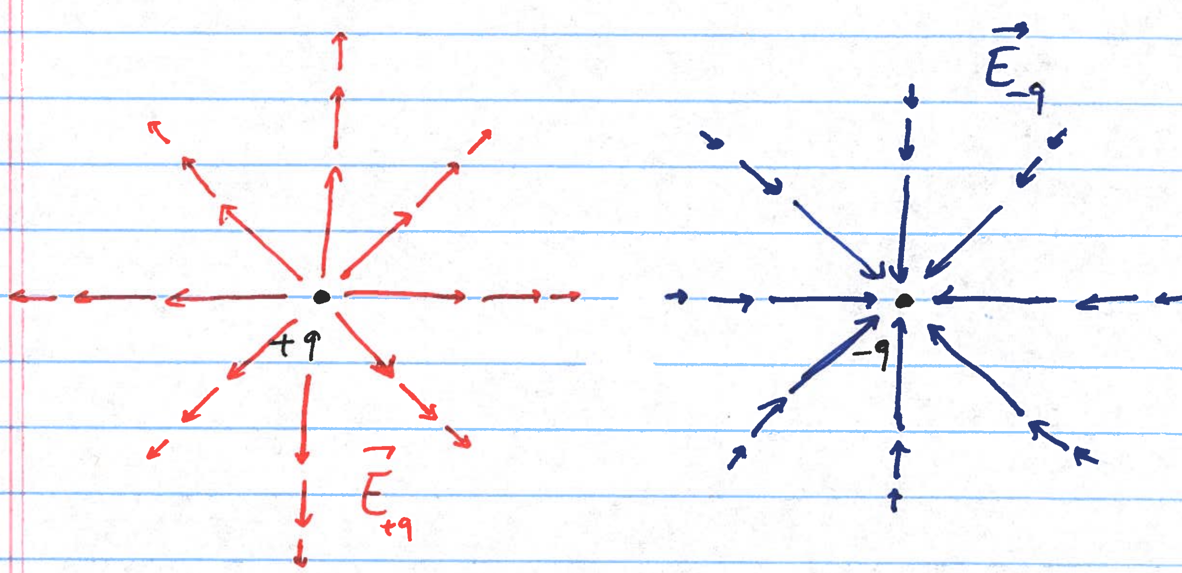
Electric field of ~~a collec~~ N point charges



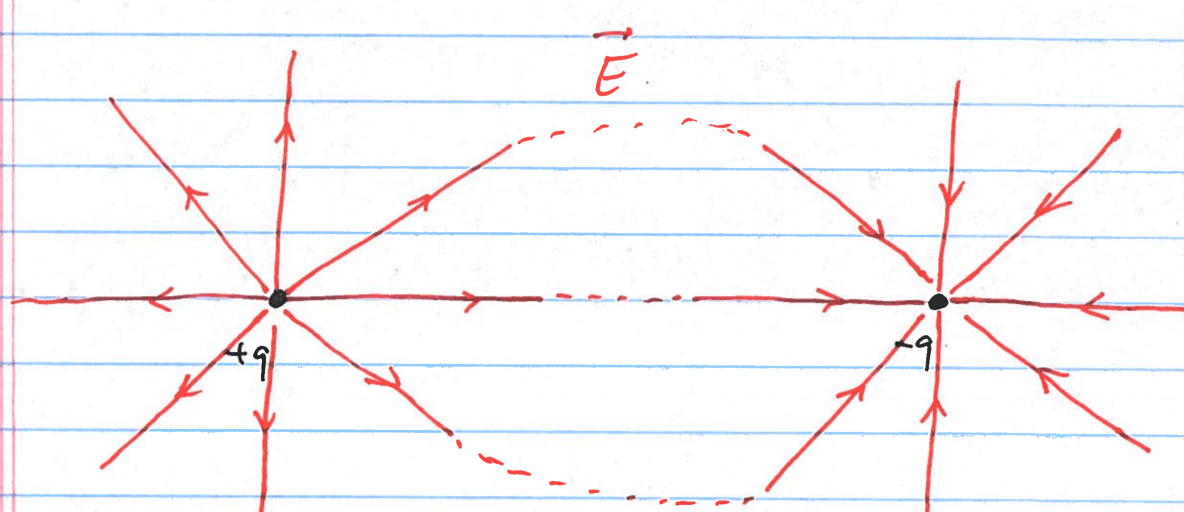
$$\vec{E}_{total}(\vec{r}) = \sum_{i=1}^{N} \vec{E}_{q_i}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{R}_i|^2} \underbrace{(\widehat{\vec{r} - \vec{R}_i})}_{\hat{r}_i}$$

Visualizing the electric field of a point charge

method 1: vector ~~int~~ / arrow indicates direction and magnitude of $\vec{E}$.



method 2: use continuous "field lines", which indicate the direction of the electric field. The magnitude of $\vec{E}$ is given by the density of the field lines



note: field lines can ~~stop~~ only end ~~on a nega~~ or begin on a charge.

negative

positive

Pillars of Electrostatics

- Electric field of a point charge falls off like $\frac{1}{r^2}$

  ↳ follows naturally in a model where $\vec{E}$-field emanates uniformarily from the point charge (in 3D)

- Superposition principle: $\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 + \ldots$

  The electric field (i.e. the Coulomb force) at a point is the $\underbrace{\text{sum}}_{\text{linear sum}}$ of electric fields from all sources

→ See Powerpoint / PDF presentation

Continuous charge distributions

electric field: $$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{\infty} \frac{q_i \, (\widehat{\vec{r}-\vec{R}_i})}{|\vec{r}-\vec{R}_i|^2} \longrightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all}\\\text{charges}}} \left[\frac{dq \, (\widehat{\vec{r}-\vec{r}\,'})}{|\vec{r}-\vec{r}\,'|^2} \right]$$

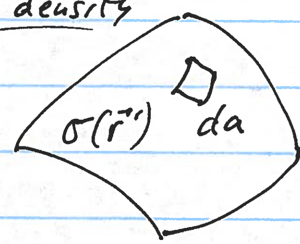
$$= \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all}\\\text{charges}}} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dq$$

for line charge density $\lambda(\vec{r}')$ $\quad dq = \lambda\, dl'$ $\quad dl'$

$$dq = \lambda(\vec{r}')\, dl'$$

e.g. dx'

for surface charge density



$$dq = \sigma(\vec{r}')\, da$$

e.g. $dx'dy'$

or

$r'dr'd\phi'$, $r'd\phi'dz'$

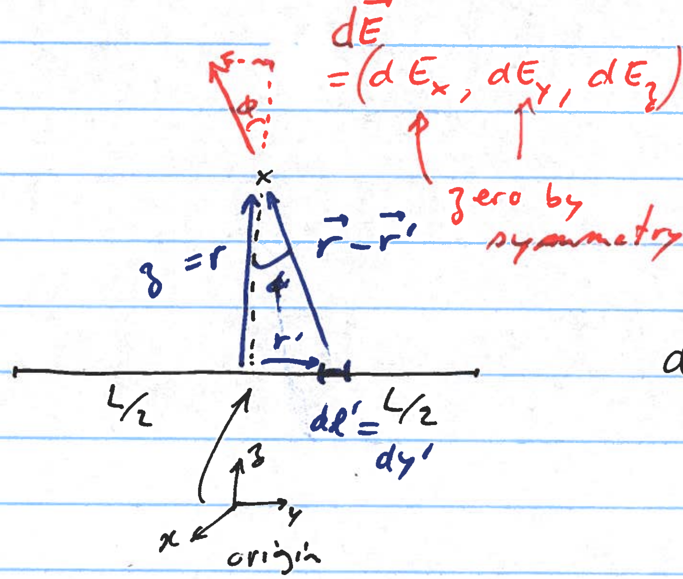
or $r'^2 \sin\theta' d\phi' d\theta'$

for volume charge density $\quad dq = \rho(\vec{r}')\, dV$

e.g. $dx'dy'dz'$

example: Electric field from a line of charge of length L and total charge q (uniformly distributed).

$$\lambda = q/L$$



$$dE_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dy' \cos\phi}{\left(\sqrt{z^2 + y'^2}\right)^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dy'}{z^2 + y'^2} \frac{z}{\sqrt{z^2 + y'^2}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dy'\, z}{(z^2 + y'^2)^{3/2}}$$

$$\vec{E} = E_z \hat{z} = \frac{1}{4\pi\varepsilon_0} \lambda z \int_{-L/2}^{L/2} \frac{dy'}{(z^2+y'^2)^{3/2}}$$

trigonometric substitution

$y' = z\tan\varphi$

$dy' = z\sec^2\varphi\, d\varphi$

$$= \frac{1}{4\pi\varepsilon_0} \lambda z \left. \frac{y'}{z^2\sqrt{z^2+y'^2}} \right|_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda z}{z^2} \underbrace{\left[\frac{L/2}{\sqrt{z^2+(L/2)^2}} - \frac{(-L/2)}{\sqrt{z^2+(L/2)^2}} \right]}_{\frac{L}{\sqrt{z^2+(L/2)^2}}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{z} \frac{\lambda L}{\sqrt{z^2+(L/2)^2}} \quad q$$

$$\Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{z\sqrt{z^2+(L/2)^2}}}$$

note: if $z \gg L$, then $\vec{E} \rightarrow \frac{1}{4\pi\varepsilon_0} \frac{q}{z^2}$

line of charge behaves like a point charge if you are ~~far~~ away from it.