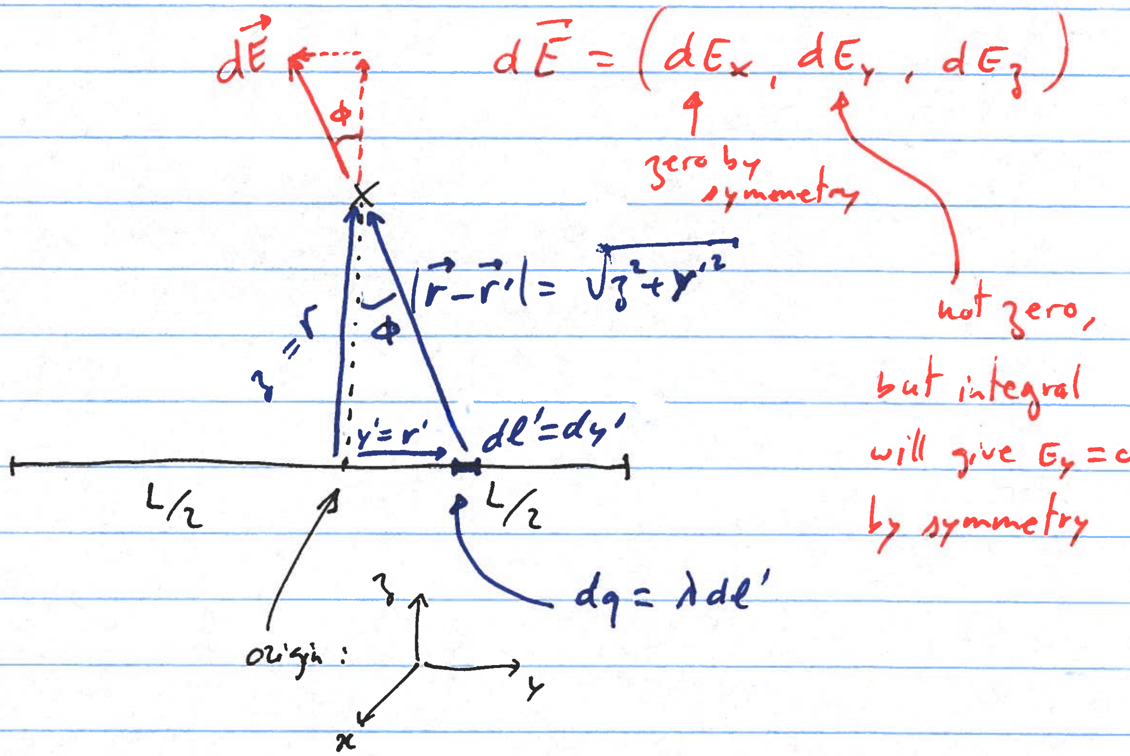
Monday, February 15, 2021

Continuous charge distributions (continued) [chpt 2.1.4 - 2.2.3]

Example 1: Electric field from a line of charge of length L and total charge q (uniformily distributed).

$\hookrightarrow \lambda = q/L$ = line charge distribution



$$dE_z = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dy'}{\left(\sqrt{z^2+y'^2}\right)^2} \cos\phi = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dy'}{\left(\sqrt{z^2+y'^2}\right)^2} \frac{z}{\sqrt{z^2+y'^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{z\lambda\, dy'}{\left(z^2+y'^2\right)^{3/2}}$$

$$\vec{E} = E_z \hat{z} = \frac{1}{4\pi\varepsilon_0} \lambda z \int_{-L/2}^{L/2} \frac{dy'}{(z^2+y'^2)^{3/2}}$$

trigonometric substitution

$y' = z\tan\varphi$

$dy' = z\sec^2\varphi\, d\varphi$

$$= \frac{1}{4\pi\varepsilon_0} \lambda z \left. \frac{y'}{z^2\sqrt{z^2+y'^2}} \right|_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda \cancel{z}}{z^{\cancel{2}}} \underbrace{\left[\frac{L/2}{\sqrt{z^2+(L/2)^2}} - \frac{(-L/2)}{\sqrt{z^2+(-L/2)^2}} \right]}_{\frac{L}{\sqrt{z^2+(L/2)^2}}}$$

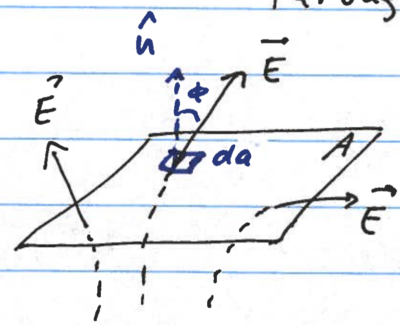
$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{z} \frac{\lambda L}{\sqrt{z^2+(L/2)^2}} \quad \lambda L = q$$

$$\Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{z\sqrt{z^2+(L/2)^2}}\, \hat{z}}$$

note: If $z \gg L$, then $\vec{E} \to \frac{1}{4\pi\varepsilon_0} \frac{q}{z^2}$

i.e. a line of charge behaves like a point charge if you are far enough away from it.

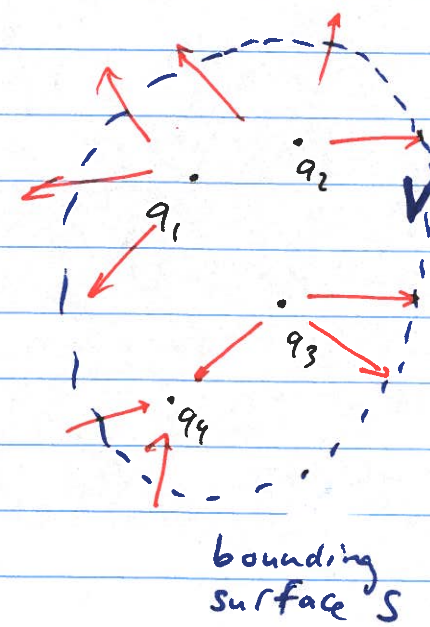
Electric flux = "flow" of the electric field through a surface A.



electric flux $\Phi = \int_A \vec{E}\cdot d\vec{a}$
(through area A)

Gauss's law

Consider a collection of charges inside a volume V with bounding surface S.



$$\vec{E}_{total}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} \frac{q_i}{|\vec{r}-\vec{r}_i|^2}\,(\widehat{\vec{r}-\vec{r}_i})$$

Electric flux through S:

$$\Phi_S = \oint_S \vec{E}_{total}\cdot d\vec{s}$$

divergence theorem

$$= \int_V (\vec{\nabla}\cdot\vec{E}_{total})\,d^3r$$

$$= \frac{1}{\cancel{4\pi}\varepsilon_0} \int_V \sum_{i=1}^{N} q_i \underbrace{\vec{\nabla}_i \cdot \frac{\widehat{(\vec{r}-\vec{r}_i)}}{|\vec{r}-\vec{r}_i|^2}}_{\cancel{4\pi}\,\delta^3(\vec{r}-\vec{r}_i)} d^3r$$

$$= \frac{1}{\varepsilon_0} \sum_{i=1}^{N} q_i \underbrace{\int_V \delta^3(\vec{r}-\vec{r}_i)\, d^3r}_{=1}$$

$$= \frac{1}{\varepsilon_0} \underbrace{\sum_{i=1}^{N} q_i}_{\text{enclosed charge in } V}$$

$$= \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\Longrightarrow \boxed{\oint_S \vec{E}_{total} \cdot d\vec{s} = \frac{Q_{enclosed}}{\varepsilon_0}}$$

Gauss's law (integral form)

Physical interpretation:

Each charge "emits" a fixed number of E-field lines ~~EF~~ (number is proportional to individual charge q_i). Since the field lines cannot terminate in free space (or cross), then the number that cross the bounding surface (flux) give a census for the total enclosed charge.

Differential form of Gauss's law

If we consider a continuous charge distribution $\rho(\vec{r})$, then

$$Q_{Enclosed} = \int_V \rho(\vec{r})\, d^3r$$

plug into Gauss's Law

$$\underbrace{\oint_S \vec{E}_{total} \cdot d\vec{s}}_{\text{divergence theorem}} = \frac{1}{\varepsilon_0} \int_V \rho(\vec{r})\, d^3r$$

$$= \int_V (\vec{\nabla} \cdot \vec{E})\, d^3r$$

$$\Rightarrow \int_V (\vec{\nabla} \cdot \vec{E})\, d^3r = \int_V \frac{\rho(\vec{r})}{\varepsilon_0}\, d^3r$$

This equation holds for any volume V, including differentially small ones, so the integrands must be equal:

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}}$$

Gauss's law [Maxwell's 1st law]

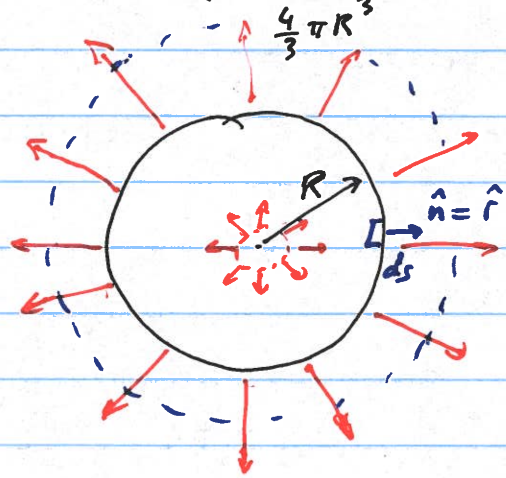
<u>physical interpretation</u>: you can only source or sink electric field "flow" (i.e. field lines) with a charge ("+" = source, "−" = sink).

Applying Gauss's law

(useful when some symmetry is present)

<u>Example 1</u>: Solid sphere of uniform charge (charge = Q, radius R)

$\hookrightarrow \rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$



$\vec{E}$ must point radially outward/inward by symmetry. $\Rightarrow \vec{E} = E\hat{r}$ $= E(r)\hat{r}$

for <u>$r > R$</u>

$$\int_S \underbrace{\vec{E} \cdot d\vec{s}}_{\substack{\vec{E} \parallel d\vec{s} \\ \Rightarrow\ = E\,ds}} = \underbrace{\frac{Q_{enclosed}}{\varepsilon_0}}_{\frac{Q}{\varepsilon_0}}$$

$$\Longrightarrow \int E\,ds = \frac{Q}{\varepsilon_0} \iff E \underbrace{\int ds}_{4\pi r^2} = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad E\, 4\pi r^2 = Q/\varepsilon_0$$

$$\Longleftrightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}}$$ for $r > R$

i.e. E-field looks like a point charge Q at origin.

for $r < R$

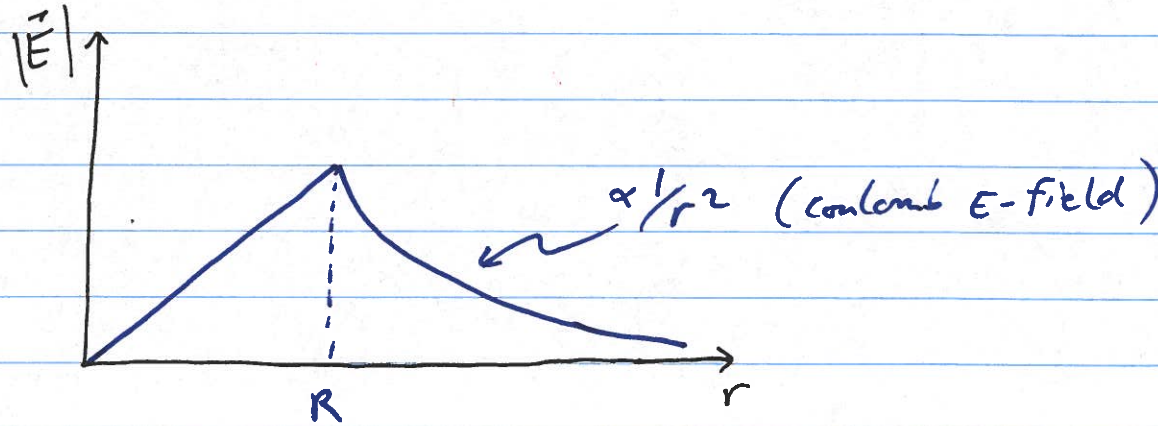
$$\underbrace{\int \vec{E}\cdot d\vec{s}}_{E\,4\pi r^2} = \frac{Q_{enclosed}}{\varepsilon_0} \longrightarrow Q_{enclosed} = \int_V \rho\, d^3r$$

$$= \frac{4}{3}\pi r^3 \rho$$

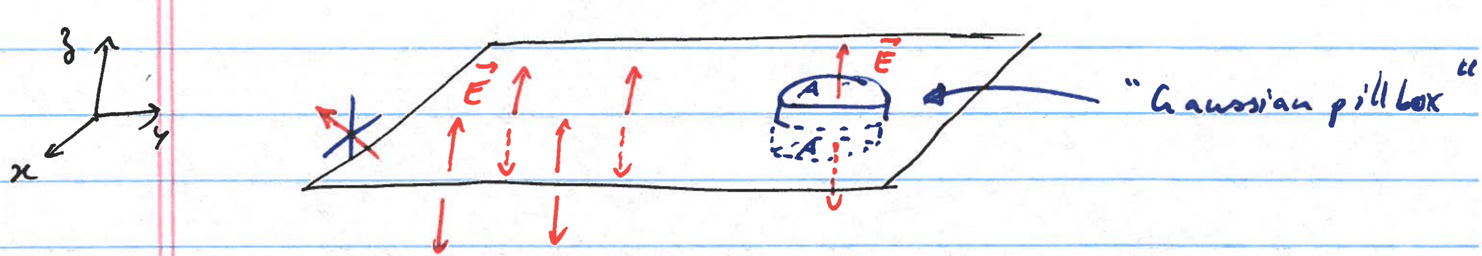
$$= \cancel{\tfrac{4}{3}\pi} r^3 \frac{Q}{\cancel{\tfrac{4}{3}\pi} R^3}$$

$$\Longrightarrow E\,4\pi \cancel{r^2} = \frac{Q}{\varepsilon_0}\frac{r^{\cancel{3}}}{R^3}$$

$$\Longrightarrow \boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3} r\,\hat{r}}$$ for $r < R$



Example 2: Infinite sheet of charge (in xy-plane) with ~~charge~~ uniform charge density σ.



Gauss's Law: $$\underbrace{\int_{S=\text{pill box}} E \cdot ds}_{} = \frac{Q_{enclosed}}{\varepsilon_0}$$ with $Q_{enclosed} = \sigma A$.

$$\underbrace{|E|A}_{\text{top surface}} + \underbrace{|E|A}_{\text{bottom surface}}$$

$$\Rightarrow \quad 2|E|\cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0}$$

$$\Rightarrow \quad |E| = \frac{\sigma}{2\varepsilon_0} \quad \Rightarrow \quad \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}}$$