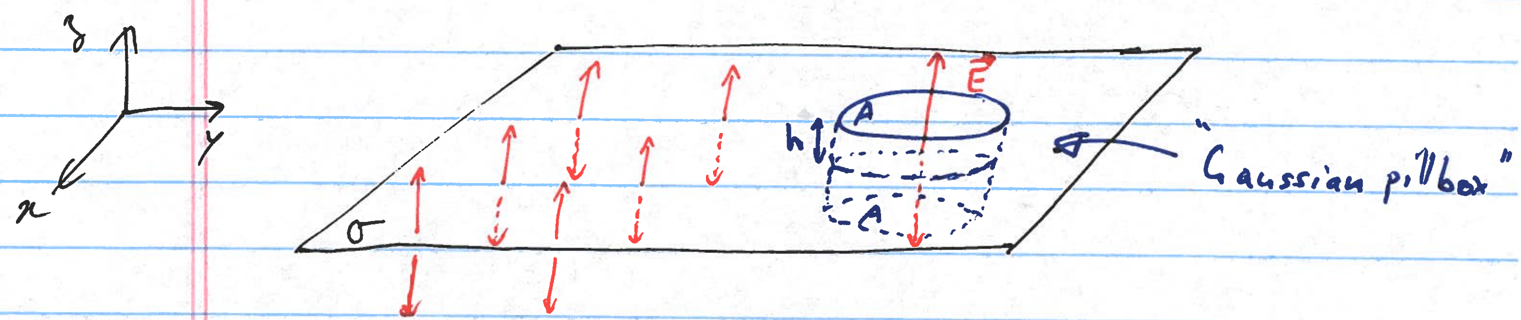
Wednesday, February 17, 2021

Applying Gauss's law (continued) [chpt. 2.2.3]

Example 1 Infinite sheet of charge (in xy-plane) with uniform charge density σ.



Gauss's law:

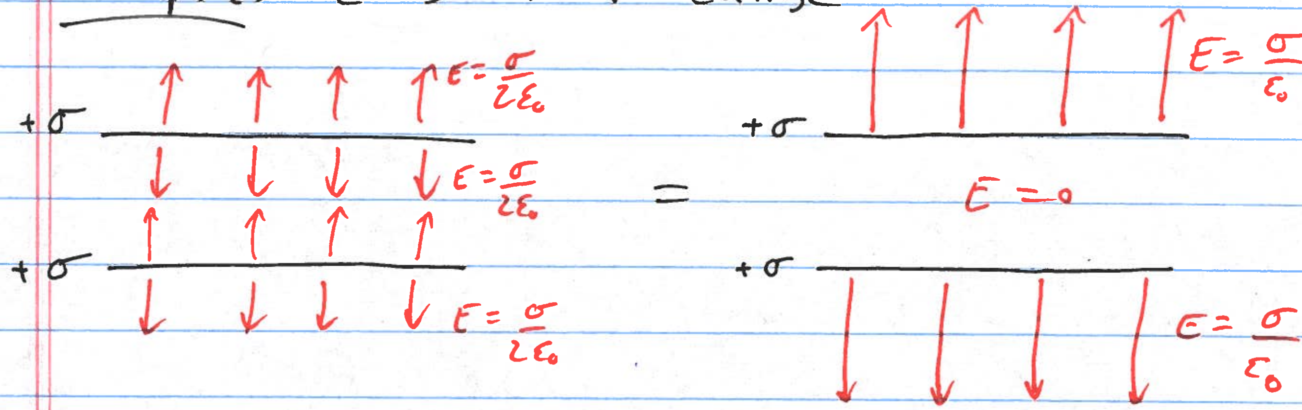
$$\underbrace{\int_{S=\text{pill box}} \vec{E}\cdot d\vec{s}}_{\underbrace{|\vec{E}|A}_{\text{top surface}} + \underbrace{|\vec{E}|A}_{\text{bottom surface}}} = \frac{\overbrace{Q_{enclosed}}^{\sigma A}}{\varepsilon_0}$$

$$\Rightarrow \quad 2|\vec{E}|A = \frac{\sigma A}{\varepsilon_0} \iff |\vec{E}| = \frac{\sigma}{2\varepsilon_0}$$

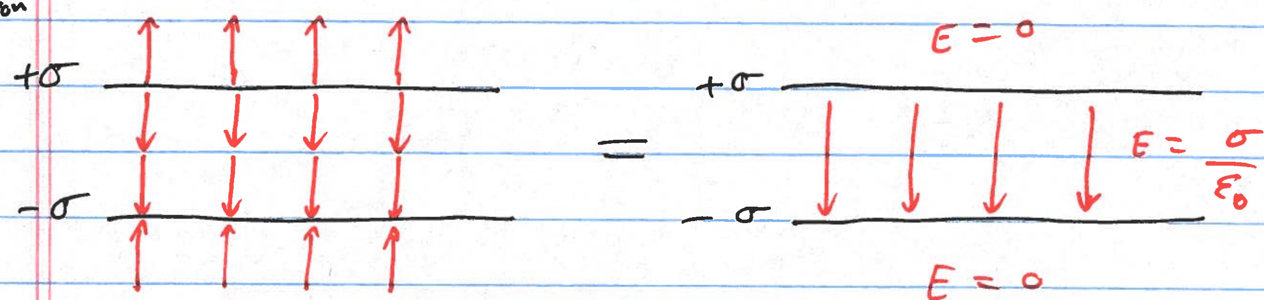
$$\Rightarrow \quad \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\,\hat{n}}$$

note: E-field is independent of distance "h" from sheet.

Example 2: 2 sheets of charge



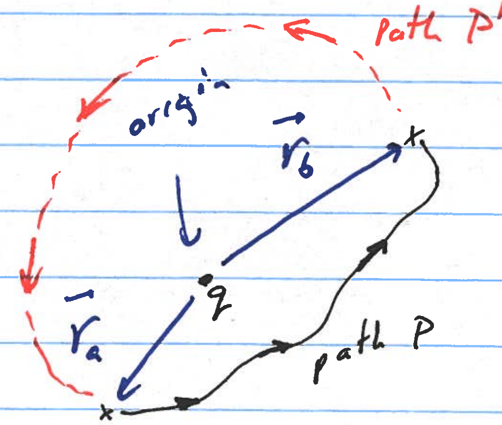
capacitor configuration



Curl of $\vec{E}$

[chpt. 2.2.4]

$$\vec{\nabla} \times \vec{E} = ?$$



Curl of a point charge (at origin)

step 1: Let's calculate $\displaystyle\int_{\vec{r}_a \text{ path } P}^{\vec{r}_b} \vec{E} \cdot d\vec{l}$

$$\vec{E}_q = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$ (spherical coordinates)

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$ (spherical coordinates)

relative size of $dr, d\theta, d\phi$ depends $d\vec{l}$ (and $\underline{r}, \theta$)

but $\vec{E}_q \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} dr$ $\quad [\hat{r}\cdot\hat{\theta} = 0,\ \hat{r}\cdot\hat{\phi} = 0]$

thus $$\underset{\text{path } P}{\int_{\vec{r}_a}^{\vec{r}_b}} \vec{E}_q \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} q \int_{r_a}^{r_b} \frac{1}{r^2} dr = \frac{1}{4\pi\varepsilon_0} q \left(-\frac{1}{r}\right)_{r_a}^{r_b}$$

$$= \frac{1}{4\pi\varepsilon_0} q \left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$\Rightarrow$ this integral depends only on the endpoints, not the path!!!

Thus $$\oint_{\text{path } P+P'} \vec{E}_q \cdot d\vec{l} = \underbrace{\int_{\vec{r}_a}^{\vec{r}_b}}_{\text{path } P} \vec{E}_q \cdot d\vec{l} + \underbrace{\int_{\vec{r}_b}^{\vec{r}_a}}_{\text{path } P'} \vec{E}_q \cdot d\vec{l} = 0$$

result does not depend on P & P'
only endpoints

$$\Rightarrow \boxed{\oint_{\text{any path}} \vec{E}_q \cdot d\vec{l} = 0}$$

apply Stokes's theorem

$$\Longleftrightarrow \quad \int_{\substack{\text{any surface} \\ \text{(with any bounding line path)}}} (\vec{\nabla} \times \vec{E}_q) \cdot d\vec{s} = 0$$

Since the result is true for any surface, even an infinitesimal one, then

$$\vec{\nabla} \times \vec{E}_q = 0$$

Since any E-field that we can have can be built by adding up the contributions of point charges, then by the superposition principle we have

$$\boxed{\vec{\nabla} \times \vec{E} = 0}$$

(true for any static electric field)

$$\Longleftrightarrow \quad \boxed{\int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{l} \text{ is independent of path.}}$$

Electric Potential [chpt 2.3]

definition: The potential $V(\vec{r})$ with respect to a reference point $\vec{r}_0$ is given by

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{l} \qquad \text{[units volts]}$$

(any path)

note: often $\vec{r}_0$ is taken at $|\vec{r}_0| \to +\infty$ (but not always)

⚠ $V(\vec{r})$ depends on the reference point $\vec{r}_0$.

⚠ the potential difference between 2 points is independent of $\vec{r}_0$.

proof:

$$V(\vec{r}_b) - V(\vec{r}_a) = -\int_{\vec{r}_0}^{\vec{r}_b} \vec{E}\cdot d\vec{l} - \left[-\int_{\vec{r}_0}^{\vec{r}_a} \vec{E}\cdot d\vec{l}\right]$$

$$= -\left\{\int_{\vec{r}_a}^{\vec{r}_0} \vec{E}\cdot d\vec{l} + \int_{\vec{r}_0}^{\vec{r}_b} \vec{E}\cdot d\vec{l}\right\}$$

$$= -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{l}$$

Connection between $\vec{E}$ & V

$$-\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{l} = V(\vec{r}_b) - V(\vec{r}_a) = \int_{\vec{r}_a}^{\vec{r}_b} (\vec{\nabla} V)\cdot d\vec{l} \qquad \text{Gradient theorem}$$

Since this equality holds for any $\vec{r}_a$ & $\vec{r}_b$, then

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

note 1: If you have $\vec{E}$, then get V with $V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$ (pick a path!)

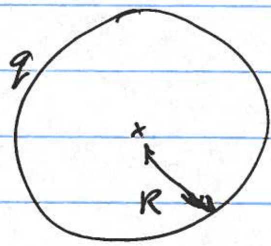
If you have V, then get $\vec{E}$ with $\vec{E} = -\vec{\nabla} V$

note 2: V obeys the superposition principle: $V_{total} = V_{q_1} + V_{q_2} + \ldots$

<u>note 3</u>: In practice, it's often more efficient to calculate $V(\vec{r})$ directly, since it has only one component (instead of 3 components for $\vec{E} = E_x, E_y, E_z$) (no vectors!)

<u>Example</u>: Potential of a uniformily charged spherical shell of radius R and charge q.

$\hookrightarrow \sigma = \dfrac{q}{4\pi R^2}$



From <u>Gauss's law</u>:

<u>$r > R$</u> : $\vec{E} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r^2} \hat{r}$

<u>note:</u> $\vec{E}$ is discontinuous at shell boundary $\Leftarrow$

<u>$r < R$</u> : $\vec{E} = 0$

(enclosed charge is zero)

<u>reference point</u>: $\vec{r}_0$ is at infinity ($|\vec{r}_0| \to +\infty$)

$$V(\vec{r}) = -\int_{\vec{r}_0\ \text{path}}^{\vec{r}} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{r} E\, dr'$$

choose a purely radial path

$$= -\int_{\infty}^{r>R} \frac{1}{4\pi\varepsilon_0} q \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0} q \left. \frac{1}{r} \right|_{\infty}^{r<R} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

thus $\displaystyle V_{r>R}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$ (also the potential for a point charge $[R\to 0]$)

for $r<R$, then we have

$$V(\vec{r}) = -\int_\infty^R \vec{E}\cdot d\vec{l} - \int_R^r \vec{E}\cdot d\vec{l}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \underbrace{\int_R^r \vec{0}\cdot d\vec{l}}_{=0}$$

$$\Rightarrow V_{r<R}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

$$\Rightarrow \boxed{\begin{aligned} V_{r>R}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \\ V_{r<R}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\frac{q}{R} \end{aligned}}$$

<u>note</u>: $V(\vec{r})$ is continuous across shell boundary.