Monday, February 21, 2021

Poisson's Equation [chpt 2.3.3]

Since $\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$ and $\vec{E} = -\nabla V$, then

$$\Rightarrow \quad \vec{\nabla}\cdot(-\vec{\nabla}V) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\Leftrightarrow \quad \boxed{\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}}$$

Poisson's equation

If no charge is present in a given volume, then

$$\boxed{\nabla^2 V(\vec{r}) = 0}$$

Laplace's equation

note 1: $\vec{\nabla}\times\vec{E} = 0$ does not give any additional information, since $\vec{\nabla}\times(-\nabla V) = -\underbrace{\vec{\nabla}\times(\vec{\nabla}V)}_{\text{curl-grad is always zero.}} = 0$

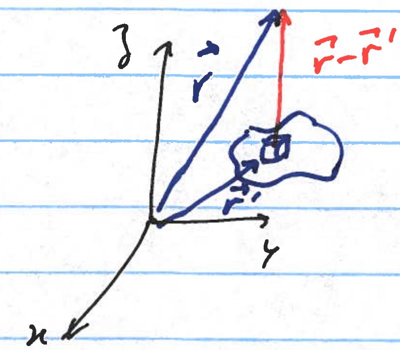
note 2: The reference point is unspecified for these equations.

↳ the reference point is implicitly defined by the boundary conditions.

i.e. where $V = 0$ (i.e. the "ground")

For a point charge: $V(\vec r) = \dfrac{1}{4\pi\varepsilon_0}\,\dfrac{q}{r}$

$$\Rightarrow \quad V(\vec r) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq(\vec r\,')}{|\vec r - \vec r\,'|}$$



$\lambda(\vec r\,')dr'$ or $\sigma(\vec r\,')d^2r'$

$$\Rightarrow \quad \boxed{V(\vec r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec r\,')\, d^3r'}{|\vec r - \vec r\,'|}}$$

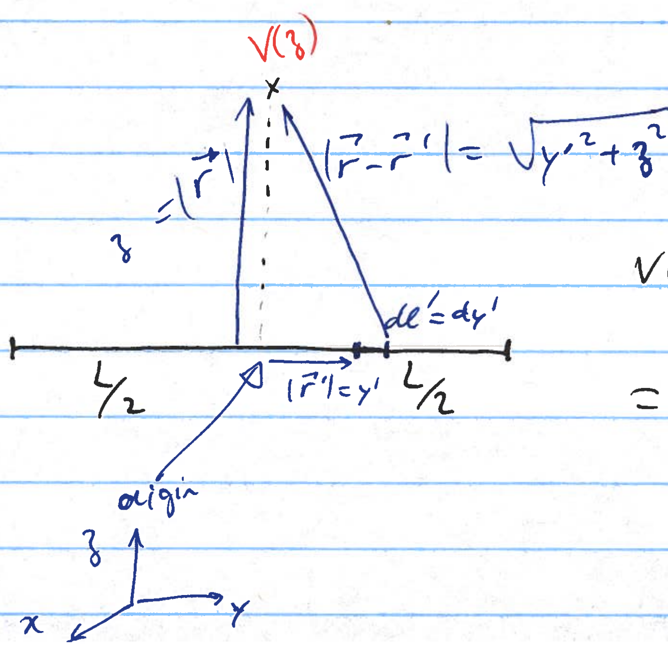
for an arbitrary charge distribution (very useful)

⚠ assumes that reference point is at $|\vec r_0| \to +\infty$

⚠ only works if you know $\rho(\vec r\,')$!!!

Example: line of charge of length L and charge q.

$$\lambda = q/L$$



$$V(\vec r) = V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2} \frac{\lambda\, dl'}{\sqrt{z^2 + y'^2}}$$

$$V(0,0,z) = V(z) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{dy'}{\sqrt{z^2+y'^2}}$$

trigonometric substitution

$y' = z \tan\varphi$

$dy' = z \sec^2\varphi \, d\varphi$

$$= \frac{\lambda}{4\pi\varepsilon_0} \, \frac{1}{2}\left[\ln\left(1+\frac{y'}{\sqrt{z^2+y'^2}}\right) - \ln\left(1-\frac{y'}{\sqrt{z^2+y'^2}}\right)\right]_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0} \, \frac{q}{\cancel{2}L} \left\{ \ln\left(1+\frac{L/2}{\sqrt{z^2+(L/2)^2}}\right) - \ln\left(1-\frac{L/2}{\sqrt{z^2+(L/2)^2}}\right) \right.$$

$$\left. \cancel{- \ln\left(1-\frac{L/2}{\sqrt{z^2+(L/2)^2}}\right)} \cancel{+ \ln\left(1+\frac{L/2}{\sqrt{z^2+(L/2)^2}}\right)} \right\}$$

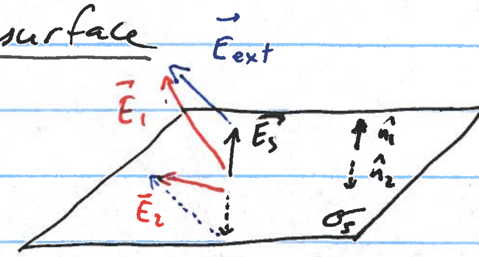
$$\boxed{V(0,0,z) = \frac{1}{4\pi\varepsilon_0} \, \frac{q}{L} \left[\ln\left(1+\frac{L/2}{\sqrt{z^2+(L/2)^2}}\right) - \ln\left(1-\frac{L/2}{\sqrt{z^2+(L/2)^2}}\right)\right]}$$

<u>note</u>: $E_z = -\frac{\partial}{\partial z} V(0,0,z) = \frac{1}{4\pi\varepsilon_0} \, \frac{q}{z\sqrt{z^2+(L/2)^2}}$

page of algebra

$\vec{E}$ & V at a charged surface

$$\vec{E}_s = \frac{\sigma_s}{2\varepsilon_0}\,\hat{n}$$



If we apply an external E-field $\vec{E}_{ext}$, then

$$\vec{E}_{total} = \vec{E}_{ext} + \vec{E}_s \Rightarrow \begin{cases} \vec{E}_1 = \vec{E}_{ext} + \dfrac{\sigma_s}{2\varepsilon_0}\,\hat{n}_1 \\ \vec{E}_2 = \vec{E}_{ext} + \dfrac{\sigma_s}{2\varepsilon_0}\,\hat{n}_2 = \vec{E}_{ext} - \dfrac{\sigma_s}{2\varepsilon}\,\hat{n}_1 \end{cases}$$

$$\Rightarrow \vec{E}_1 - \vec{E}_2 = \boxed{\Delta\vec{E}\Big|_s = \frac{\sigma_s}{\varepsilon_0}\,\hat{n}_1}$$

$$\Rightarrow \boxed{\begin{aligned} E_{1\perp} - E_{2\perp} &= \frac{\sigma_s}{\varepsilon_0} \\ E_{1//} - E_{2//} &= 0 \end{aligned}}$$

$\Rightarrow$ normal component is discontinuous

$\Rightarrow$ parallel component is continuous

potential: $V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\ell$ $\Rightarrow$ V is from an integral

path P

$\Rightarrow$ V is continuous across any surface.

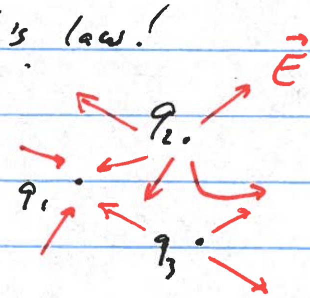
note: $\dfrac{\partial V}{\partial n} = -\dfrac{\sigma}{\varepsilon_0}$

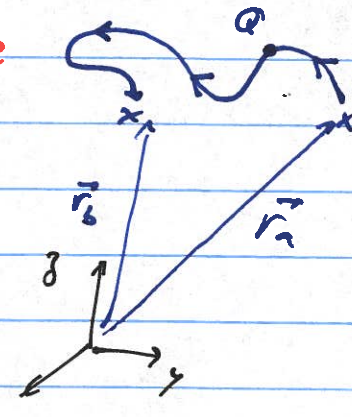
$\dfrac{\partial V}{\partial n} = (\vec{\nabla} V)\cdot\hat{n}$

Energy

<u>Q</u>: Given the electric field $\vec{E}(\vec{r})$ generated by some charge distribution, how much <u>work</u> must be done to move ~~the ch~~ a test charge Q from $\vec{r}_a$ to $\vec{r}_b$?
(along some path)

<u>A</u>: Integrate Coulomb's law!
(along path)





force provided by you (counter coulomb force)

$$\text{Work} = W = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}\cdot d\vec{\ell} = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}_{\text{coulomb}} \cdot d\vec{\ell}$$

$$= -\int_{\vec{r}_a}^{\vec{r}_b} Q\, E(\vec{r})\cdot d\vec{\ell} = -Q\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r})\cdot d\vec{\ell}$$

$$\Rightarrow \boxed{W = Q\left[V(\vec{r}_b) - V(\vec{r}_a)\right]}$$

(independent of path)
↳ potential is conservative

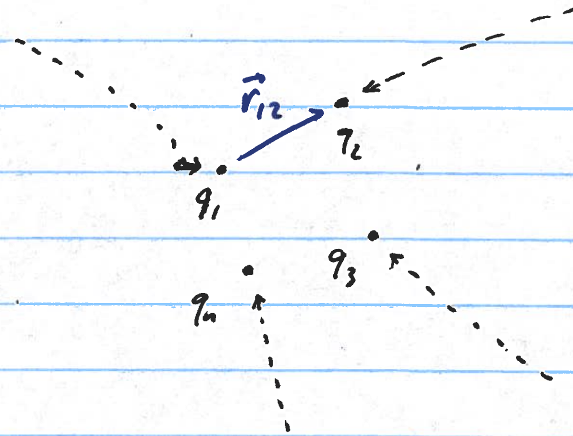
$\Rightarrow$ the potential is the <u>work</u> per unit charge.

<u>note</u>: if $|\vec{r}_a| \to +\infty$ (with $V(r\to\infty) = 0$), then

$$\boxed{W = Q\,V(\vec{r})}$$

Energy of a charge distribution (i.e. energy to assemble a charge distribution)

point charges:



$W_1 = 0$ (no other charges yet)

$$W_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}}$$

$$W_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

$$\Rightarrow \text{total work} = W = W_1 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

for n charges

$$\Rightarrow \text{total work} = W$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\underline{\underline{j>i}}}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{1}{2} \sum_{j\neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j \neq i}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{ij}} \right)}_{\text{potential at } \vec{r}_i \text{ due to all charges that are } \underline{\underline{\text{not}}}\ q_i}$$

$r_{ij} = |\vec{r}_i - \vec{r}_j|$

$$\Rightarrow W = \frac{1}{2} \sum_{i=1}^{n} q_i \, V(\vec{r}_i)$$

continuous charge distribution (generalization)

$$W = \frac{1}{2} \int_V \rho(\vec{r}\,') \, V(\vec{r}\,') \, d^3 r'$$

V: volume of charge density (or bigger)

⚠ note: now we are counting the ~~charge at $\vec{r}_i$~~ contribution to the potential from charge q_i