Wednesday, February 24, 2021

Energy of a continuous charge distribution

Work to assemble charges (i.e. charge density)

$$W = \frac{1}{2}\int_V \rho(\vec{r}')\, V(\vec{r}')\, d^3r' \qquad \text{also } \rho(\vec{r}') = \varepsilon_0 \vec{\nabla}\cdot\vec{E}(\vec{r}')$$

Volume of charge density (or bigger)

$$W = \frac{1}{2}\int_V \varepsilon_0 \underbrace{(\vec{\nabla}\cdot\vec{E})\, V}_{\vec{\nabla}\cdot(V\vec{E}) - \vec{E}\cdot\vec{\nabla}V}\, d^3r'$$

(from inside front cover of Griffiths "eq. 5")

$$= \frac{\varepsilon_0}{2}\left\{\underbrace{\int_V \vec{\nabla}\cdot(V\vec{E})\, d^3r'}_{\text{divergence theorem: } \int_S V\vec{E}\cdot d\vec{s}} + \int_V \vec{E}\cdot\underbrace{(-\vec{\nabla}V)}_{\vec{E}}\, d^3r'\right\}$$

$$= \frac{\varepsilon_0}{2}\left\{\underbrace{\int_S \underset{\frac{1}{r'}}{V}\, \underset{\frac{1}{r'^2}}{\vec{E}} \cdot \underset{r'^2}{d\vec{s}}}_{\propto \frac{r'^2}{r'^3} = \frac{1}{r'} \;=0 \text{ for } r'\to+\infty} + \int_V \vec{E}^2\, d^3r'\right\}$$

for $V \to$ very large

↳ size of universe

$\vec{E}(r'\to+\infty) \propto \frac{1}{r'^2}$

$V(r'\to\infty) \propto \frac{1}{r'}$

$$\Rightarrow \boxed{W = \frac{\varepsilon_0}{2}\int_{\text{all space}} \vec{E}^2(\vec{r}')\, d^3r'}$$

always positive

↳ includes work needed to "make" point charges.

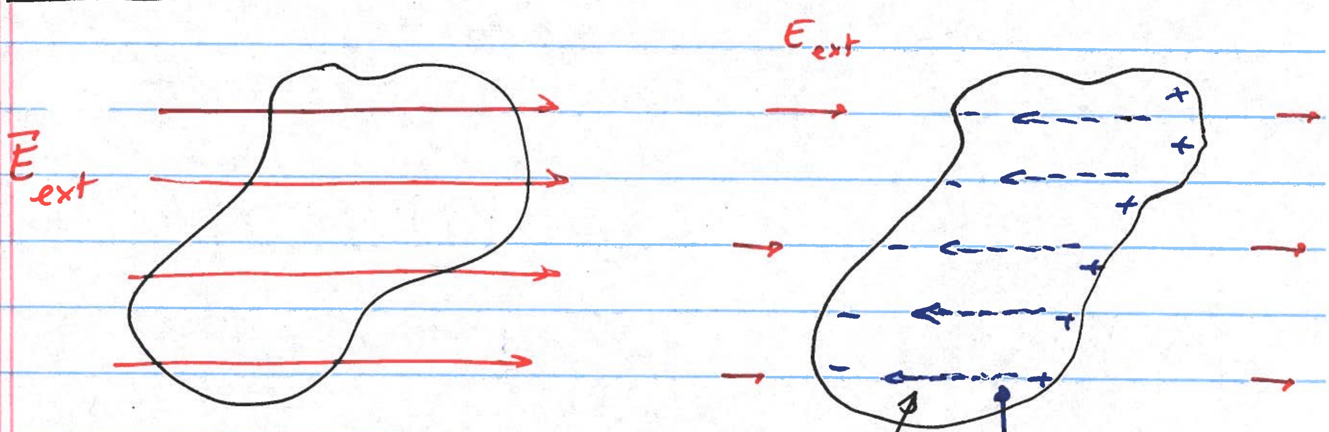
↳ this is the energy stored in the electric field

energy density $\boxed{\mathcal{E}(\vec{r}) = \frac{\varepsilon_0}{2}\vec{E}^2(\vec{r})}$ electric energy per unit volume

Perfect Conductors (e.g. metals such as Cu, Al, Au, Ar, etc...)

definition: a body with an unlimited supply of free charges (positive and negative) that are free to move around within ~~the~~ its volume and on its surface.

property #1: Inside a conductor $\vec{E} = 0$



before: charges re-arrange themselves

very very fast "instantaneous" for PHYS 401

$\vec{E}_{\substack{total\\inside}} = 0$ $\vec{E}_{charges} = -\vec{E}_{ext}$

after charges re-arrange themselves

$\Rightarrow \boxed{\vec{E}_{\substack{inside\\conductor}} = \vec{0}}$

[if it wasn't zero then the free charges would be pushed and thus re-arrange themselves]

$\Rightarrow V = \text{constant}_{\substack{inside\\conductor}}$

↳ conductor is an equipotential.

Property #2: Inside a conductor $\rho(\vec{r}) = 0$

$$\vec{\nabla} \cdot \vec{E}_{inside} = 0 = \rho_{inside}(\vec{r})/\varepsilon_0 \Rightarrow \boxed{\rho_{inside}(\vec{r}) = 0}$$

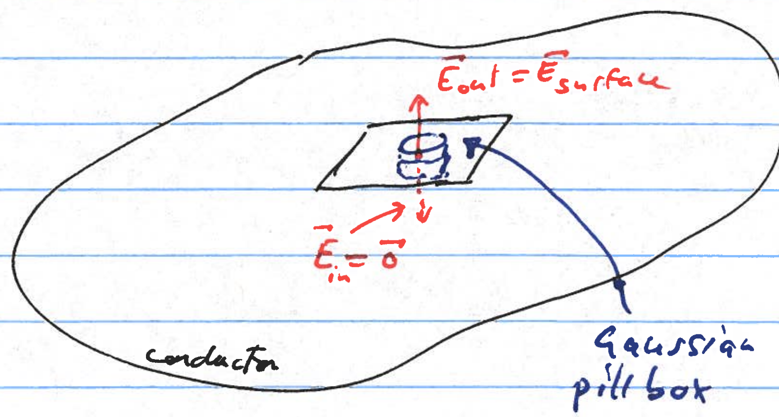
Property #3:

$\Rightarrow$ All charge resides on the surface of the conductor [includes charges from conductor + any added free charges]

property #4: The E-field just outside the conductor is perpendicular to its surface with

$$\boxed{\vec{E}_{surface} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

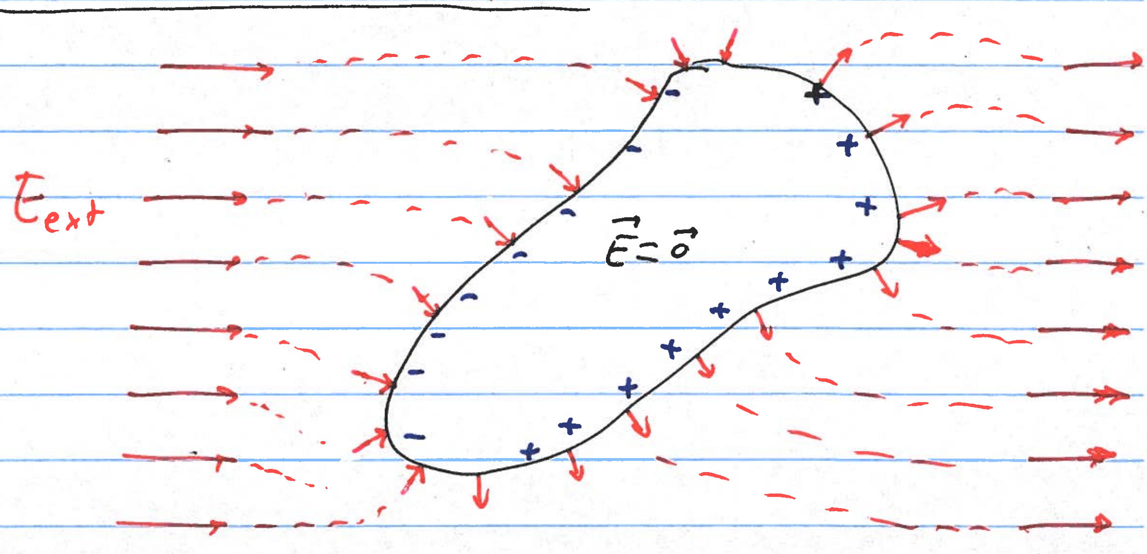
note: If $\vec{E}_s$ had a parallel component, then charge would flow and redistribute itself.



$$E_{out} A = \frac{1}{\varepsilon_0}\sigma A$$

$$\Rightarrow \vec{E}_{out} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

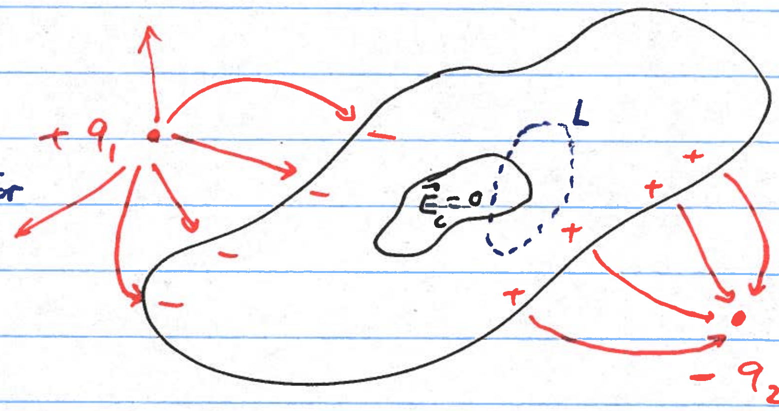
Illustration: Conductor in an external E-field



Screening and shielding

Case 1: Consider a conducting volume V with an empty cavity inside V_c inside of it. $\vec{E}_c$ inside the cavity is zero for any arrangements of charges/fields outside of V (and V_c)

note: q_1 & q_2 are attracted to conductor



proof that $\vec{E}_c = 0$: Suppose that $\vec{E}_c \neq 0$, then consider a loop L that is in the conductor and cavity.

$$\vec{\nabla} \times \vec{E} = \vec{0} \Rightarrow \oint_L \vec{E} \cdot d\vec{l} = 0 \Leftrightarrow \int_{L_c} \vec{E}_c \cdot d\vec{l} + \int_{L_v} \cancel{\vec{E}_v}^{\,0} \cdot d\vec{l} = 0$$

$$\Rightarrow \int_{L_c} \vec{E}_c \cdot d\vec{l} = 0 \quad \text{for any } L_c$$

$$\Rightarrow \vec{E}_c = 0$$

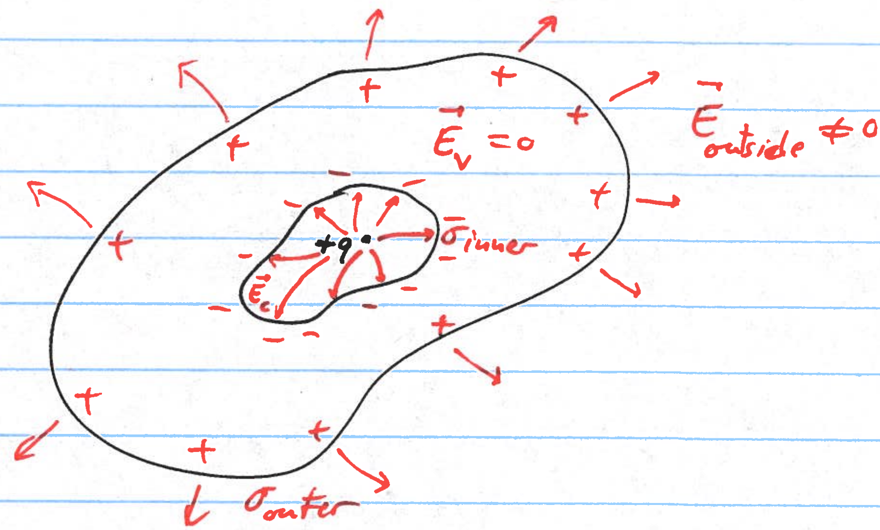
Physics proof: Consider the conductor without the cavity, the $\vec{E}_c = \vec{0}$. Now add in the cavity: no free charge is added and σ_{outer} stays the same, thus $\vec{E}_c = \vec{0}$

Corollary: $\sigma_c = 0 \Rightarrow$ there is no surface charge on the cavity surfaces (otherwise $\Delta\vec{E}\big|_{\text{surface of } c} \neq 0$)

$\Rightarrow$ shielding electric fields is easy $\Rightarrow$ just make a conducting box (i.e. Faraday cage)

↳ in practice, a conducting mesh is frequently sufficient.

Case 2 : Consider a charge $+q$ inside the cavity.
In this case, $\vec{E}_c \neq \vec{0}$ and $\vec{E}_{outside} \neq \vec{0}$



σ_{inner} cancels the effect of the charge $\underline{q}$ inside the conductor so that $\vec{E}_V = \vec{0}$ (inside conductor).

↳ thus σ_{inner} depends on the cavity surface and the position of q.

σ_{outer} is distributed so as to not produce an E-field inside the conductor on its own (i.e. ~~else~~ independently of σ_{inner}) ⟶ σ_{outer} depends only on the outer surface of V (and the magnitude of $\underline{q}$).

↳ argument: remove charge $+q$ from cavity, and add total charge $+q$ to the outer surface of V (lots of little charge)

↳ the charges will arrange themselves to make σ_{outer} in which $\vec{E}_V = 0$ (solution exists)

[in this case $\vec{E}_c = 0$ and $\sigma_{inner} = 0$.]