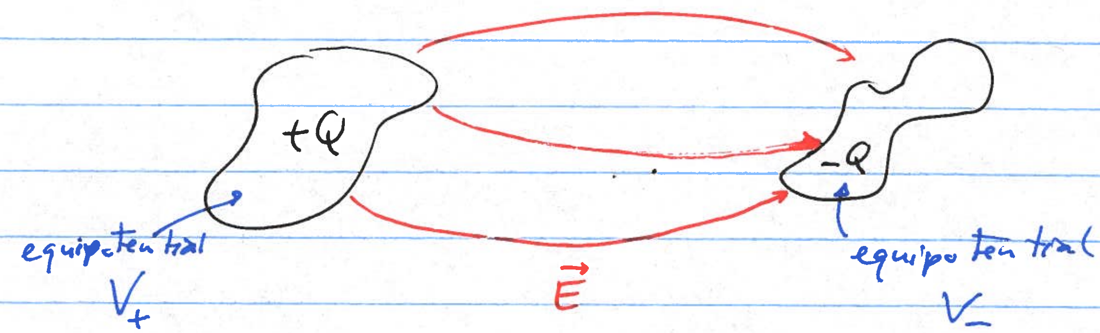
Monday, March 1, 2021

Capacitors [chpt.

Consider two conductors with charge $+Q$ and $-Q$.



The E-field strength is proportional to Q] $\vec{E} \propto Q$
the potential difference is proportional to Q] $\Delta V = V_+ - V_- \propto Q$

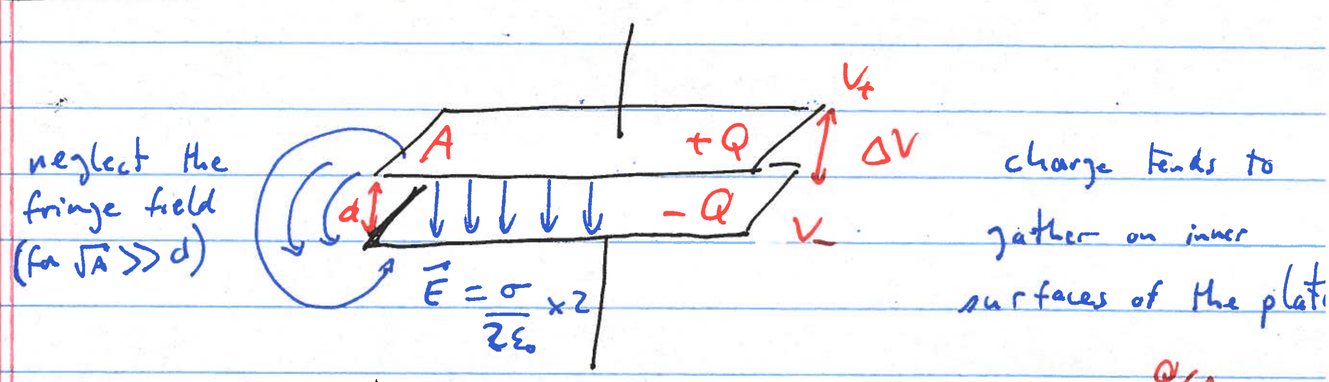
The capacitance C is the proportionality constant between ΔV and Q:

$$C = \frac{Q}{\Delta V} = \text{Coulombs per volt}$$

$[C\Delta V = Q]$

Capacitance is a purely geometric quantity.

Example: parallel plate capacitor.



$$\Delta V = -\int_{V_- \text{ plate}}^{V_+ \text{ plate}} \vec{E} \cdot d\vec{l} = Ed \Rightarrow \vec{E} = \frac{\Delta V}{d} = 2 \times \frac{\sigma}{2\varepsilon_0}$$

Q/A

potential difference between plates — charge on single plate

$$\Rightarrow \frac{\Delta V}{d} = \frac{Q/A}{\varepsilon_0} \iff \frac{\varepsilon_0 A}{d}\Delta V = Q$$

$$\Rightarrow \boxed{C = \frac{\varepsilon_0 A}{d}}$$

Energy stored in a capacitor

If you start with an uncharged capacitor, the work to move charge from one conductor/terminal to the other is

$$dW = \underbrace{V}_{q/C}\, dq \Rightarrow W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{C}\frac{q^2}{2}\Big|_0^Q$$

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\underbrace{\left(\frac{Q}{C}\right)^2}_{V}$$

$$\Rightarrow \boxed{W = \text{stored energy} = \frac{1}{2}CV^2}$$

$$= \frac{1}{2}\frac{Q^2}{C}$$

The energy is stored in the electric field of the capacitor

$\Rightarrow$ anything that has an electric field has a capacitance

$\Rightarrow$ you can also calculate capacitance C with

$$\boxed{\frac{\varepsilon_0}{2}\int \vec{E}^2 d^3r = \frac{1}{2}CV^2}$$

note: $\text{Energy} = \frac{\varepsilon_0}{2}\int \vec{E}^2 d^3r$

$$= \frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 Ad = \frac{\sigma^2}{2\varepsilon_0}Ad \qquad \sigma = \frac{Q}{A}$$

$$= \frac{Q^2}{A^2}\frac{Ad}{2\varepsilon_0} \Rightarrow \boxed{\text{Energy} = \frac{Q^2}{2\varepsilon_0}\frac{d}{A}}$$

$\text{Energy} = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$

$$= \frac{1}{2}\frac{Q^2}{\left(\frac{\varepsilon_0 A}{d}\right)} =$$

$$\Rightarrow \boxed{\text{Energy} = \frac{Q^2}{2\varepsilon_0}\frac{d}{A}}$$

Calculating Electric Fields and Potentials [chpt 3]

(I) Laplace's equation: Uniqueness theorems [chpt 3.1.5, 3.1.6]

In a region of space without charges, the electric potential $V(\vec{r})$ must satisfy Laplace's equation:

$$\boxed{\nabla^2 V(\vec{r}) = 0 \iff \frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0}$$

note: solving this equation (+ boundary conditions) can be quite difficult.

Uniqueness Theorem #1

The solution to $\nabla^2 V(\vec{r}) = 0$ in some volume $\mathbb{V}$ is uniquely determined if $V(\vec{r})$ is specified on the boundary surface $S(\mathbb{V})$.

Alternate statement: The potential $V(\vec{r})$ in a volume $\mathbb{V}$ is uniquely determined if (a) the charge density throughout $\mathbb{V}$, and (b) the value of $V(\vec{r})$ on all boundaries, are specified.

Solution Strategy:

or solution form

If you can guess a solution that satisfies $\vec{\nabla}^2 V = 0$ and the boundary conditions, then it must be the correct solution

Uniqueness theorem #2

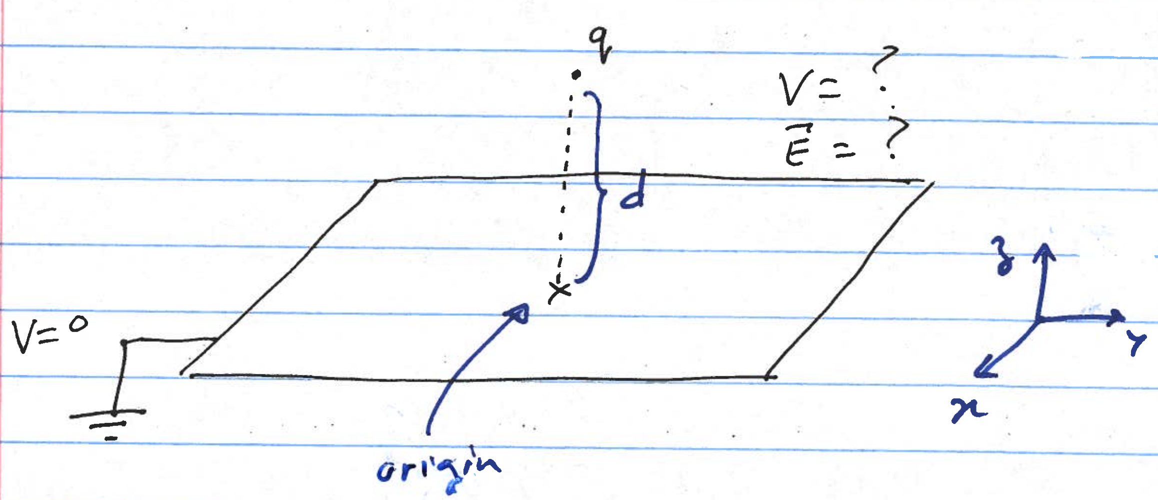
In a volume $\mathbb{V}$ surrounded by conductors (or extending to infinity) and containing a specified charge density $\rho(\vec{r})$, the E-field is uniquely determined if the total charge on each conductor is given.

the charge on the conductors can move around.

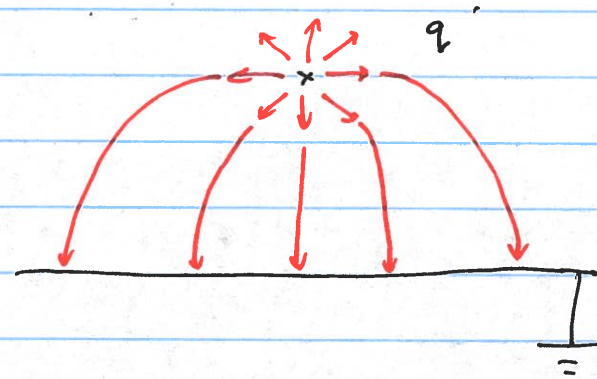
II Method of Images

Standard Example: A point charge q is held a distance d above an infinite grounded (i.e. $V_{conductor} = 0$) conducting plane.

Q: What is the potential $V(\vec{r})$ and E-field $\vec{E}(\vec{r})$ in the region above the plane?



<u>qualitative guess</u>:

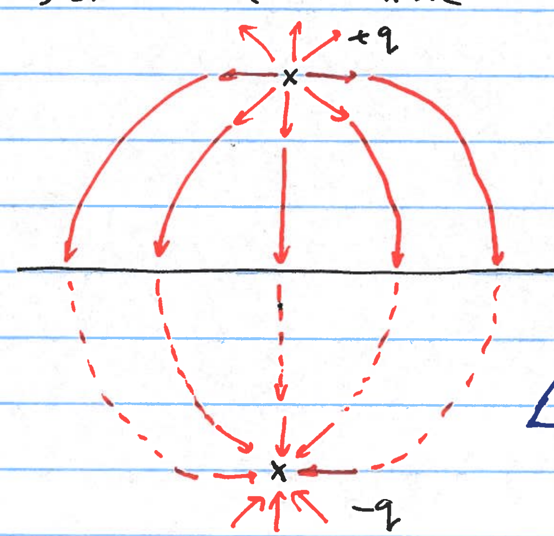


$\vec{E}_{plane} \perp$ plane

near the point charge: $\vec{E}_q \simeq \dfrac{1}{4\pi\varepsilon_0} \dfrac{q \, (x, y, z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}$

$$V_q \simeq \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{x^2+y^2+(z-d)^2}}$$

The solution sort of looks like



$\vec{E}_{plane} \perp$ plane $(z=0)$

⚠ there are lots of possible ways for $\vec{E}_{plane}$ to vary over the plane.

quantitative guess:

possible solution: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$

we note : $\begin{cases} V(z=0) = 0 \\ V(r\to+\infty) = 0 \qquad (\text{for } z>0) \end{cases}$

$\hookrightarrow$ These are the boundary conditions and they are satisfied.

$\boxed{\Rightarrow \text{ this must be the solution by the uniqueness theorem (\#1)}}$

- the solution is only valid for $z>0$.
- The solution also works if the $z<0$ region is replaced by the conductor.

Q: What's the surface charge density?

$$\Delta\vec{E} = \vec{E}_{z>0} - \underbrace{\vec{E}_c}_{=0} = \underline{\sigma}\,\hat{n} \iff \vec{E}_{z>0} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\Rightarrow -\nabla V = \frac{\sigma}{\varepsilon_0}\hat{n} \iff -\frac{\partial V}{\partial z}\bigg|_{z=0}\hat{z} = \frac{\sigma}{\varepsilon_0}\hat{z}$$

$$\frac{\partial V}{\partial x}\bigg|_{z=0} = \frac{\partial V}{\partial y}\bigg|_{y=0} = 0$$