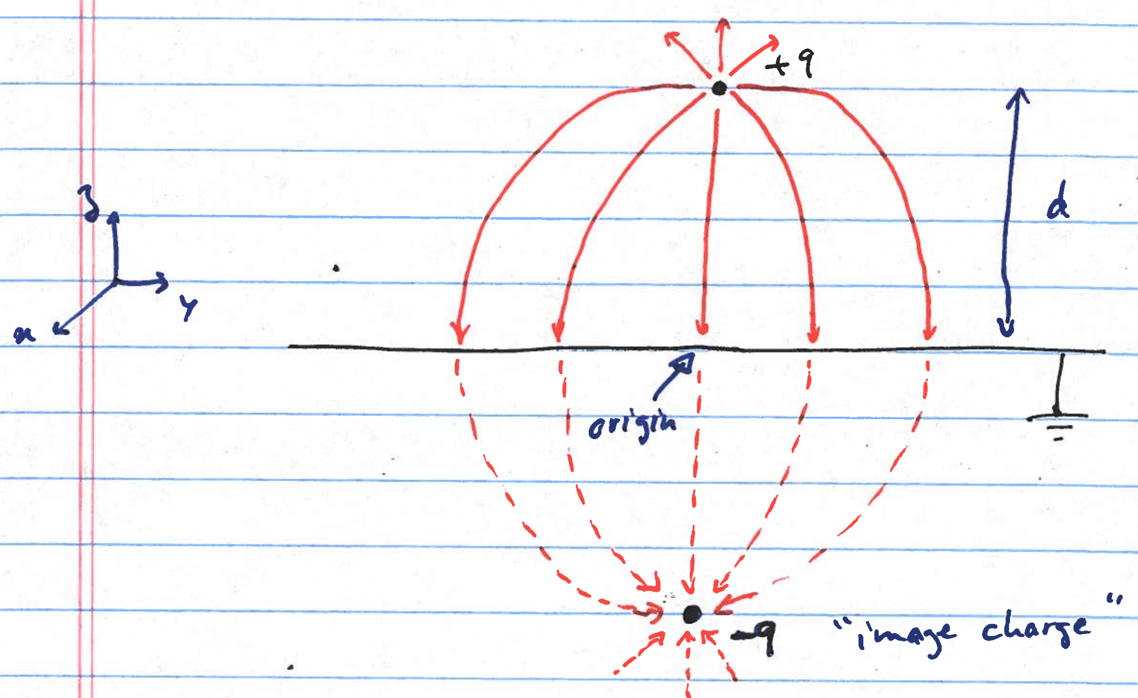
Wednesday, March 3, 2021

METHOD OF Images (continued)

point charge above an infinite conducting grounded plane.



Solution: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\overbrace{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}^{\text{charge}} - \overbrace{\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}}^{\text{image charge potential}}\right]$

↳ satisfies b.c. $V(z=0)=0$, $V(r\to+\infty)=0$ | the solution is only valid in the region without the image charge.

Q: What's the surface charge density?

$$\Delta E_{surface} = \vec{E}_{z>0} - \underbrace{\vec{E}_c}_{=0} = \frac{\sigma}{\varepsilon_0}\hat{n} \Leftrightarrow \vec{E}(z=0^+) = \frac{\overset{\sigma(x,y)}{\sigma}}{\varepsilon_0}\hat{n}$$

$$\Rightarrow -\nabla V\Big|_{z=0^+} = \frac{\sigma}{\varepsilon_0}\hat{n} \Leftrightarrow -\frac{\partial V}{\partial z}\Big|_{z=0^+}\hat{z} = \frac{\sigma}{\varepsilon_0}\hat{z} \quad \left[\frac{\partial V}{\partial x}\Big|_{z=0^+} = \frac{\partial V}{\partial y}\Big|_{z=0^+} = 0\right]$$

#2

$$\left.\frac{\partial V}{\partial x}\right|_{z=0} = \frac{1}{4\pi\varepsilon_0} q\left[-\frac{1}{2}\frac{2x}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \left(-\frac{1}{2}\right)\frac{2x}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right]$$

$$= 0 \qquad \text{also} \quad \left.\frac{\partial V}{\partial y}\right|_{z=0} = 0$$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{1}{4\pi\varepsilon_0} q\left[\left(-\frac{1}{\cancel{2}}\right)\frac{\cancel{2}(\cancel{z}-d)}{\left[x^2+y^2+(\cancel{z}-d)^2\right]^{3/2}} - \left(-\frac{1}{\cancel{2}}\right)\frac{\cancel{2}(\cancel{z}+d)}{\left[x^2+y^2+(\cancel{z}+d)^2\right]^{3/2}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2d}{\left[x^2+y^2+d^2\right]^{3/2}}$$

since $\left.\frac{\partial V}{\partial z}\right|_{z=0} = -\frac{\sigma}{\varepsilon_0} \Rightarrow \frac{\sigma}{\varepsilon_0} = -\frac{1}{4\pi\varepsilon_0} q \frac{(2d)}{\left[x^2+y^2+d^2\right]^{3/2}}$

$$\Rightarrow \boxed{\sigma = -\frac{q}{2\pi}\frac{d}{\left[x^2+y^2+d^2\right]^{3/2}}}$$

induced charge density

total induced charge $= q_c = \int \sigma \, ds$

$$= -\frac{q}{2\pi} d \int_{xy\ \text{plane}} \frac{dx\,dy}{\left[x^2+y^2+d^2\right]^{3/2}} = -\frac{qd}{2\pi}\int_0^{+\infty}\int_0^{2\pi}\frac{r\,dr\,d\phi}{\left[r^2+d^2\right]^{3/2}}$$

$$= -\frac{q}{2\pi} d\, 2\pi \int_0^{+\infty}\frac{r\,dr}{\left[r^2+d^2\right]^{3/2}} = -qd\left(\frac{-1}{\sqrt{r^2+d^2}}\right)_0^{\infty}$$

$$= -qd\left(0 - \left(-\frac{1}{|d|}\right)\right) = -q$$

Thus the induced charge = $\boxed{q_c = -q}$ (as expected)

Force on the charge q (by the conducting plane)

$$F = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{(-q)\,q}{(2d)^2}}_{\substack{\text{coulomb force}\\ \text{due to image}\\ \text{charge}}}\,\hat{z}$$

$\Rightarrow$ charge is attracted to the conducting plane.

Electrostatic energy is half that for the $\underbrace{\text{charge + image charge}}_{\text{half of universe is missing}}$

$$W = \int_{\infty}^{d} \vec{F}\cdot\underbrace{d\vec{l}}_{-dz} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}\,dz$$

$$= \frac{q^2}{4\pi\varepsilon_0}\frac{1}{4}\left.-\frac{1}{z}\right|_{\infty}^{d} = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{4}\left(-\frac{1}{d}-0\right)$$

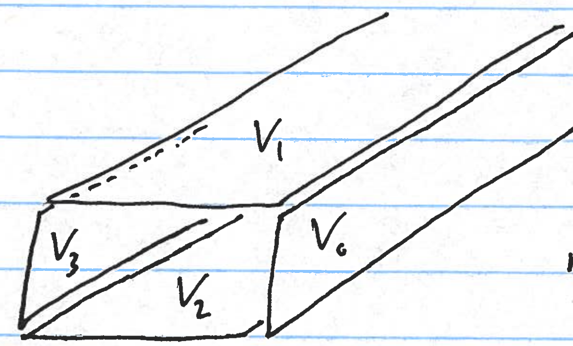
$$= -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

$$\Rightarrow \boxed{W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}}$$

Separation of Variables

Consider a system with Cartesian symmetry
(in the geometry)

Example:



rectangular duct with conducting walls
(no extra charges)

Q: What is the potential inside the duct?

no charges (except on boundaries)
$\longrightarrow$ Laplace's equation: $\nabla^2 V = 0$

$$\Leftrightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x,y,z) = 0$$

Ansatz: We will consider separable solutions of the form
only possible because of symmetry
(each coordinate gets its own function)

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \quad Y(y)Z(z)\frac{\partial^2}{\partial x^2}X + X(x)Z(z)\frac{\partial^2}{\partial y^2}Y + X(x)Y(y)\frac{\partial^2}{\partial z^2}Z = 0$$

Divide by $X(x)\,Y(y)\,Z(z)$

$$\Rightarrow \frac{1}{X(x)} \frac{\partial^2}{\partial x^2} X + \frac{1}{Y(y)} \frac{\partial^2}{\partial y^2} Y + \frac{1}{Z(z)} \frac{\partial^2}{\partial z^2} Z = 0$$

Since the solution must work for all (x, y, z) in the solution volume, then each term is equal to a constant such that

$$\Rightarrow \begin{cases} \dfrac{1}{X(x)} \dfrac{\partial^2}{\partial x^2} X = C_x & \Longleftrightarrow \quad \dfrac{\partial^2}{\partial x^2} X = C_x X = \alpha^2 X \\[2ex] \dfrac{1}{Y(y)} \dfrac{\partial^2}{\partial y^2} Y = C_y & \Longleftrightarrow \quad \dfrac{\partial^2}{\partial y^2} Y = C_y Y = \beta^2 Y \\[2ex] \dfrac{1}{Z(z)} \dfrac{\partial^2}{\partial z^2} Z = C_z & \Longleftrightarrow \quad \dfrac{\partial^2}{\partial z^2} Z = C_z Z = \gamma^2 Z \end{cases}$$

with $\quad C_x + C_y + C_z = 0 \quad \Longleftrightarrow \quad \boxed{\begin{array}{c} \alpha^2 + \beta^2 + \gamma^2 = 0 \\ \alpha, \beta, \gamma \in \mathbb{C} \end{array}}$

Solutions:

$$X_\alpha = \begin{cases} A_0 + B_0 x & \alpha = 0 \\ A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0 \end{cases}$$

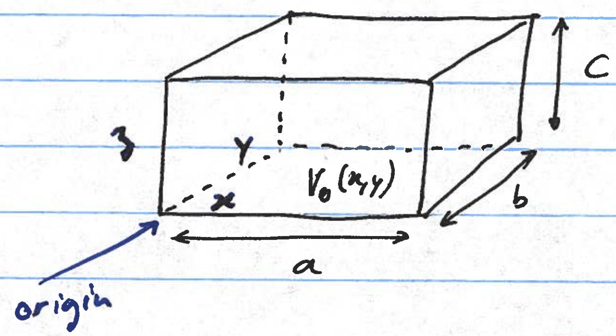
$$Y_\beta = \begin{cases} C_0 + D_0 y & \beta = 0 \\ C_\beta e^{\beta y} + D_\beta e^{-\beta y} & \beta \neq 0 \end{cases}$$

$$Z_\alpha = \begin{cases} E_0 + F_0 z & \gamma = 0 \\ E_\gamma e^{\gamma z} + F_\gamma e^{-\gamma z} & \gamma \neq 0 \end{cases}$$

<u>General Solution</u>: We consider any linear superposition of these solutions.

$$V(x,y,z) = \sum_{\alpha,\beta,\alpha} X_\alpha(x)\, Y_\beta(y)\, Z_\gamma(z) \underbrace{\delta(\alpha^2+\beta^2+\gamma^2)}_{\text{enforces } \alpha^2+\beta^2+\gamma^2=0}$$

<u>Example</u>: Consider a rectangular conducting box with 5 sides at potential $V=0$ and one side at potential $V_0(x,y)$.



<u>boundary conditions</u>:

$$\begin{cases} X_\alpha(x=0)=0 \\ X_\alpha(x=a)=0 \end{cases} \qquad \begin{cases} Y_\beta(y=0)=0 \\ Y_\beta(y=b)=0 \end{cases}$$

$$Z_\gamma(z=c)=0 \qquad \text{and } V(x,y,0)=V_0(x,y)$$

easier harder

<u>For X_α</u>:

$$X_\alpha(x=0)=0 \Rightarrow A_\alpha + B_\alpha = 0 \iff A_\alpha = -B_\alpha$$

$$X_\alpha(x=a)=0 \Rightarrow A_\alpha e^{\alpha a} - A_\alpha e^{-\alpha a} = 0$$

$$\iff e^{\alpha a} = e^{-\alpha a}$$

$$\iff e^{2\alpha a} = 1 \Rightarrow 2\alpha a = 0 \Rightarrow \alpha = 0$$

or

$$2\alpha a = n i\, 2\pi$$

$$\Rightarrow \boxed{\alpha = \frac{i n\pi}{a}}$$

with $n = 0, \pm 1, \pm 2, \ldots$