Monday, March 8, 2021

Summary of last class:

For an electric system with Cartesian symmetry (in the geometry), we consider separable solutions.

$V(x,y,z) = X(x)\,Y(y)\,Z(z)$ satisfies $\nabla^2 V(x,y,z) = 0$

where

$$\begin{cases} X_\alpha(x) = A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0 \\ Y_\beta(y) = C_\beta e^{\beta y} + D_\beta e^{-\beta y} & \beta \neq 0 \\ Z_\gamma(z) = E_\gamma e^{\gamma z} + F_\gamma e^{-\gamma z} & \gamma \neq 0 \end{cases}$$

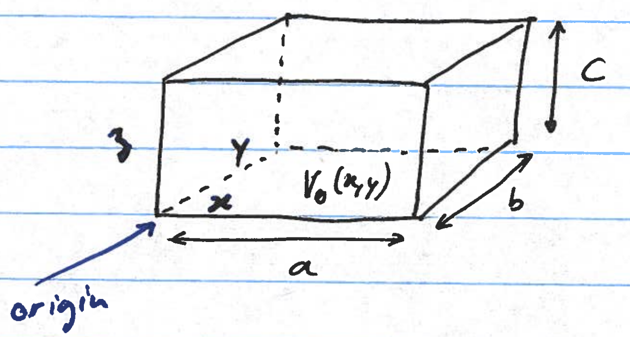
and $\alpha^2 + \beta^2 + \gamma^2 = 0$

General Solution: We consider any linear superposition of these solutions.

$$V(x,y,z) = \sum_{\alpha,\beta,\alpha} X_\alpha(x)\, Y_\beta(y)\, Z_\gamma(z)\ \underbrace{\delta(\alpha^2+\beta^2+\gamma^2)}_{\text{enforces } \alpha^2+\beta^2+\gamma^2=0}$$

superposition coefficient are inside $X_\alpha, Y_\beta, Z_\gamma$

Example: Consider a rectangular conducting box with 5 sides at potential $V=0$ and one side at potential $V_0(x,y)$.



boundary conditions:

$$\left\{\begin{array}{l} X_\alpha(x=0)=0 \\ X_\alpha(x=a)=0 \end{array}\right. \qquad \left\{\begin{array}{l} Y_\beta(y=0)=0 \\ Y_\beta(y=b)=0 \end{array}\right.$$

$$Z_\gamma(z=c)=0 \qquad \text{and} \quad V(x,y,0)=V_0(x,y)$$

easier   harder

For X_α:

$$X_\alpha(x=0)=0 \Rightarrow A_\alpha + B_\alpha = 0 \Leftrightarrow A_\alpha = -B_\alpha$$

$$X_\alpha(x=a)=0 \Rightarrow A_\alpha e^{\alpha a} - A_\alpha e^{-\alpha a} = 0$$

$$\Leftrightarrow e^{\alpha a} = e^{-\alpha a}$$

$$\Leftrightarrow e^{2\alpha a} = 1 \Rightarrow 2\alpha a = 0 \Rightarrow \alpha = 0$$

or

$$2\alpha a = n\, i\, 2\pi$$

$$\Rightarrow \boxed{\alpha = \frac{i n \pi}{a}}$$

with $n = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow X_\alpha(x) = 2i A_\alpha \left[\frac{e^{i(n\pi/a)x} - e^{-i\left(\frac{n\pi}{a}\right)x}}{2i} \right]$$

$$\Rightarrow \boxed{X_n(x) = A_n \sin\left(\frac{n\pi}{a} x\right)} \quad \text{with } n = 0, \pm 1, \pm 2, \ldots$$

For Y_β: Similarly, $\boxed{Y_m = C_m \sin\left(\frac{m\pi}{b} y\right)}$ and $\boxed{\beta = \frac{im\pi}{b}}$

with $m = 0, \pm 1, \pm 2, \ldots$

For Z_γ: $Z_\gamma(z=c) = 0 \Rightarrow \underbrace{E_\gamma}_{G/2} e^{\gamma c} + \underbrace{F_\gamma}_{-G/2} e^{-\gamma c} = 0$

$$\Rightarrow \begin{cases} E_\gamma = e^{-\gamma c} \frac{G_\gamma}{2} \\ F_\gamma = -e^{+\gamma c} \frac{G_\gamma}{2} \end{cases}$$

without loss of generality

$$\Rightarrow Z_\gamma(z) = G_\gamma \left[\frac{e^{-\gamma c}}{2} e^{\gamma z} - \frac{e^{\gamma c}}{2} e^{-\gamma z} \right]$$

$$= G_\gamma \left[\frac{e^{\gamma(z-c)} - e^{-\gamma(z-c)}}{2} \right]$$

$$= G_\gamma \sinh\left[\gamma(z-c)\right]$$

$$\Rightarrow Z_\gamma(z) = G_\gamma \sinh\left[\gamma(z-c)\right] \quad \text{with } \gamma^2 = -(\alpha^2 + \beta^2)$$

$$= \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

$$\Rightarrow \boxed{Z_{mn}(z) = G_{mn} \sinh\left[\gamma_{mn}(z-c)\right]}$$

$$\Rightarrow \boxed{\gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}$$

Thus

$$V(x,y,z) = \sum_{m,n=1}^{\infty} \underbrace{G_{mn} A_n C_m}_{\text{rename } V_{mn}} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \sinh\left[\gamma_{mn}(z-c)\right]$$

Find boundary condition: $V(x,y,z=0) = \underbrace{V_0(x,y)}_{\text{not yet specified/given}}$

$$\Rightarrow V_0(x,y) = \sum_{m,n=1}^{\infty} V_{mn} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh\left[\gamma_{mn}(\cancel{z}^{\,0}-c)\right]$$

$$= -\sum_{m,n=1}^{\infty} V_{mn} \sinh(\gamma_{mn} c) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right)$$

$\Rightarrow$ This is a Fourier series representation of $V_0(x,y)$

Q: How do we determine V_{mn}?

A: $$\int_0^a \int_0^b V_0(x,y) \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{m'\pi y}{b}\right) dx\,dy$$

$$= -\sum_{m,n=1}^{\infty} V_{mn} \sinh(\gamma_{mn} c) \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{n'\pi x}{a}\right) dx \int_0^b \sin\left(\frac{m\pi}{b}y\right) \sin\left(\frac{m'\pi y}{b}\right) dy$$

$$\boxed{\text{substitution: } u = \frac{\pi x}{a},\ du = \frac{\pi}{a}dx \quad v = \frac{\pi}{b}y,\ dv = \frac{\pi}{b}dy}$$

$$= -\sum_{m,n=1}^{\infty} V_{mn} \sinh(\gamma_{mn} c)\, \frac{a}{\pi} \underbrace{\int_0^{\pi} \sin(nu)\sin(n'u)\,du}_{=\pi/2\, \delta_{nn'}\ \text{orthogonality relation}} \times \frac{b}{\pi} \underbrace{\int_0^{\pi} \sin(mv)\sin(m'v)\,dv}_{=\pi/2\, \delta_{mm'}\ \text{orthogonality relation}}$$

$$\boxed{\text{Important: } \int_0^{\pi} \sin(nx)\sin(mx)\,dx = \frac{\pi}{2}\delta_{mn}}$$

use: $\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$\Rightarrow$ the sum dissappears!

drop the "the primes"

$$= -\frac{ab}{4} V_{m'n'} \sinh(\gamma_{m'n'} c)$$

$$\Rightarrow \boxed{V_{mn} = \frac{-4}{ab \sinh(\gamma_{mn} c)} \int_0^a \int_0^b V_0(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dx\,dy}$$

with $\gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$

<u>Special case</u>: $V_0(x,y) = V_0$

$$V_{mn} = \frac{-4V_0}{ab \sinh(\gamma_{mn} c)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx \int_0^b \sin\left(\frac{m\pi y}{b}\right) dy$$

<u>note</u>:

$$\int_0^a \sin\left(\frac{n\pi}{a} x\right) dx = -\left(\frac{a}{n\pi}\right) \cos\left(\frac{n\pi}{a} x\right)\Big|_0^a = -\left(\frac{a}{n\pi}\right) \underbrace{\left[\cos(n\pi) - 1\right]}_{= \begin{cases} -2 & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases}}$$

$$= \begin{cases} \frac{2a}{n\pi} & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases}$$

Thus

$$V_{mn} = \frac{-4V_0}{\cancel{ab} \sinh(\gamma_{mn} c)} \left(\frac{2\cancel{a}}{n\pi}\right)\left(\frac{2\cancel{b}}{m\pi}\right) \quad \text{for } m,n = \text{odd}$$

$$= \frac{-16}{nm\pi^2} \frac{V_0}{\sinh(\gamma_{mn} c)}$$

Thus finally, we get

$$\boxed{V(x,y,z) = \sum_{\substack{m,n=1\\(\text{odd})}}^{\infty} \frac{-16}{n m \pi^2} \frac{V_0}{\sinh(\gamma_{mn} c)} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh\left(\gamma_{mn}(z-c)\right) \quad \text{with } \gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}$$

denominators grow for $n, m \to$ large

$\Rightarrow$ Solution is an infinite series.

Note: You can construct solutions for more complicated arrangements based on the above solution: