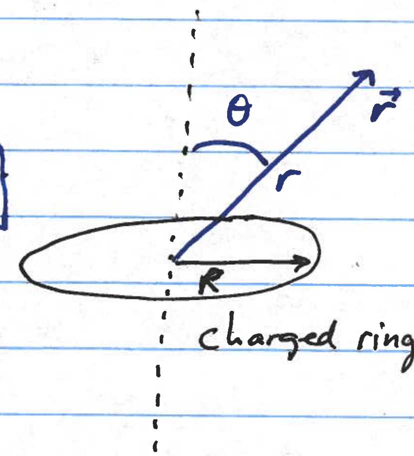
Wednesday, March 10, 2021

Separation of variables : Spherical Symmetry

Simplest case : Azimuthal Symmetry

We expect $V(r, \theta, \phi) = V(r, \theta)$



In spherical coordinates: $\boxed{\nabla^2 V(r,\theta) = 0}$

Laplace's eq.

$$\Longleftrightarrow \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)}_{} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

sometimes useful $\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)$

[multiply by r^2] If we assume $V(r,\theta) = R(r)\,\Theta(\theta)$ as the separable solution, then we get

$$\Rightarrow \Theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + R \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) = 0$$

divide by $R(r)\,\Theta(\theta)$

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)}_{\text{Constant} = C_r} + \underbrace{\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right)}_{\text{Constant} = C_\theta} = 0$$

[Constant since each term is independent of the other]

note: $C_r = -C_\theta = k(k+1)$ by convention

for $R(r)$: $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R(r)}{\partial r}\right) = k(k+1)\, R(r)$$

↳ solution: $$\boxed{R_k(r) = A_k r^k + B_k r^{-(k+1)}}$$

for $\Theta(\theta)$: $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Theta}{\partial\theta}\right) = -k(k+1)\,\Theta$$

↳ convenient change of variable: $x = \cos\theta$

↑ not coordinate x

↳ then $\dfrac{\partial}{\partial\theta} = \dfrac{\partial}{\partial x}\underbrace{\dfrac{\partial x}{\partial\theta}}_{-\sin\theta} = -\sin\theta\dfrac{\partial}{\partial x}$

Thus the differential equation becomes:

$$\underbrace{\frac{1}{\sin\theta}(-\sin\theta)}_{-1}\frac{\partial}{\partial x}\left(\underbrace{\sin\theta\,(-\sin\theta)}_{\substack{-\sin^2\theta \\ = \cos^2\theta - 1 = x^2-1}}\frac{\partial}{\partial x}\,\Theta(x)\right) = -k(k+1)\,\Theta(x)$$

$$\Leftrightarrow \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x}\Theta(x)\right] = -k(k+1)\,\Theta(x)$$

$$\Leftrightarrow \boxed{(1-x^2)\frac{\partial^2}{\partial x^2}\Theta(x) - 2x\frac{\partial}{\partial x}\Theta(x) + k(k+1)\,\Theta(x) = 0}$$

Legendre differential equation

Solution: $$\Theta(x) = C_k P_k(x) + D_k Q_k(x)$$

$$\Rightarrow \Theta(\theta) = C_k P_k(\cos\theta) + D_k Q_k(\cos\theta)$$

$$\Theta(\theta) = C_k \underbrace{P_k(\cos\theta)}_{\substack{\text{Legendre function of} \\ \text{the first kind}}} + D_k \underbrace{Q_k(\cos\theta)}_{2^{nd}\text{ kind}}$$

does not diverge over

$0 \leq \theta \leq \pi$

for $k = 0, 1, 2, 3, \ldots$

$k \to \ell =$ positive integers

$\lim_{\substack{\theta \to 0 \\ \text{or} \\ \theta \to \pi}} Q_k(\cos\theta) \to \infty$

the divergence generally eliminates this type of solution.

note: negative ℓ integers are "ok", but $P_{-\ell-1}(x) = P_\ell(x)$

Thus, generally, we only consider solutions in terms of the Legendre polynomials: $\Theta(\theta) = P_\ell(\cos\theta)$

$\ell = 0, 1, 2, 3, \ldots$

Legendre Polynomials

Rodrigues formula: $P_\ell(x) = \dfrac{1}{2^\ell \ell!}\left(\dfrac{d}{dx}\right)^\ell (x^2-1)^\ell$

$P_0(x) = 1$	even
$P_1(x) = x$	odd
$P_2(x) = (3x^2-1)/2$	even
$P_3(x) = (5x^3-3x)/2$	odd
$P_4(x) = (35x^4-30x^2+3)/8$	even
$P_5(x) = (63x^5-70x^3+15x)/8$	odd

<u>General solution</u>: $$V(r,\theta)=\sum_{\ell=0}^{\infty}\overbrace{\left(A_\ell r^{\ell}+B_\ell r^{-(\ell+1)}\right)}^{R(r)}\overbrace{P_\ell(\cos\theta)}^{\Theta(\theta)}$$

diverges for $r\to+\infty$
↳ use near $r\to 0$

diverges for $r\to 0$
↳ use for $r\to+\infty$

In order to avoid divergences at $r\to 0$ & $r\to+\infty$, we consider the solution:

$$V(r,\theta)=\begin{cases}\displaystyle\sum_{\ell=0}^{\infty} C_\ell\left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta) & \text{for } r\leq R\\ \displaystyle\sum_{\ell=0}^{\infty} C_\ell\left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta) & \text{for } r\geq R\end{cases}$$

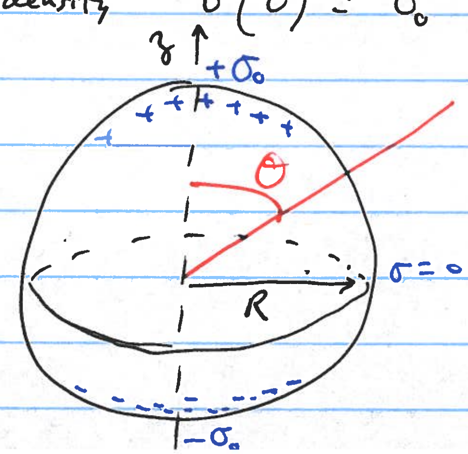
same coefficients ensures agreement at $r=R$

$\underline{R}$ can be any finite radius, but it is generally suggested by the particular problem.

⚠ If there is a boundary at $r=R$, then there may be a boundary condition to satisfy.

<u>Example</u>: Spherical shell ~~with~~ (non-conducting) with charge density $\sigma(\theta)=\sigma_0\cos\theta$ and radius R.

→ Calculate $V(r,\theta)$ everywhere (inside & outside).



<u>direct method</u>: $$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\theta')}{|\vec{r}-\vec{r}\,'|} r'^2 \sin\theta' \, dr' d\theta' d\phi'$$

↳ somewhat messy

<u>Separation of variables:</u>

<u>inside shell</u> $(r<R)$: $$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} C_\ell \left(\frac{r}{R}\right)^\ell P_\ell(\cos\theta)$$

<u>outside shell</u> $(r>R)$: $$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} C_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta)$$

[$\perp$ component is discontinuous
// component is continuous]

[see Lecture 8
Monday, Feb. 22]

<u>Boundary conditions</u>

$$E_{out,r}\Big|_{r=R} - E_{in,r}\Big|_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0} = \frac{\sigma_0\cos\theta}{\varepsilon_0}$$

$E_r = -\frac{\partial V}{\partial r}$

$$\Leftrightarrow \quad \frac{\partial V_{out}}{\partial r}\Big|_{r=R} - \frac{\partial V_{in}}{\partial r}\Big|_{r=R} = -\frac{\sigma(\theta)}{\varepsilon_0} = -\frac{\sigma_0\cos\theta}{\varepsilon_0}$$

$$\Leftrightarrow \quad \sum_{\ell=0}^{\infty} \frac{C_\ell \, \ell \, r^{\ell-1}}{R^\ell} P_\ell(\cos\theta)\Big|_{r=R} - \sum_{\ell=0}^{\infty} C_\ell \frac{R^{\ell+1}}{r^{\ell+2}}(-\ell-1) P_\ell(\cos\theta)\Big|_{r=R} = +\frac{\sigma(\theta)}{\varepsilon_0}$$

$r=R$

$$\Rightarrow \quad \sum_{\ell=0}^{\infty} \frac{C_\ell \, \ell}{R} P_\ell(\cos\theta) - \sum_{\ell=0}^{\infty} \frac{C_\ell}{R}(-\ell-1) P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

$$\Leftrightarrow \quad \sum_{\ell=0}^{\infty} \frac{C_\ell}{R}(2\ell+1) P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

<u>Q</u>: How do we determine C_ℓ's ?