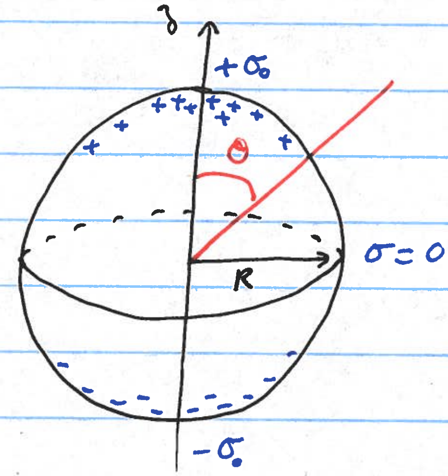
Monday, March 22, 2021

Summary from last time

Example 1: Spherical shell (non-conducting) with charge density $\sigma(\theta) = \sigma_0 \cos\theta$ and radius R.

↳ objective calculate $V(r,\theta)$ everywhere (inside & outside).



Separation of variables (spherical - azimuthal symmetry)

inside shell ($r<R$): $V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} C_\ell \left(\frac{r}{R}\right)^{\ell} \underbrace{P_\ell(\cos\theta)}_{\text{Legendre Polynomials}}$

outside shell ($r>R$): $V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} C_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta)$

From the boundary condition, we get the following condition on the coefficients: $\left[\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\Big|_{r=R} = -\frac{\sigma(\theta)}{\varepsilon_0} \right]$

$$\sum_{\ell=0}^{\infty} \frac{C_\ell (2\ell+1)}{R} P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

Q: How do we determine the C_ℓ's ?

A: Apply Fourier's trick! (for Legendre Polynomials)

↳ multiply by $P_{\ell'}(\cos\theta)$

→ integrate by $dx = d(\cos\theta)$

$$\Rightarrow \int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta)\left[\sum_{\ell=0}^{\infty} \frac{C_\ell}{R}(2\ell+1) P_\ell(\cos\theta)\right] d(\cos\theta)$$

$$= \int_{\cos\theta=-1}^{\cos\theta=+1} \frac{\sigma(\theta)}{\varepsilon_0} P_{\ell'}(\cos\theta)\, d(\cos\theta)$$

$$\Leftrightarrow \sum_{\ell=0}^{\infty} \frac{C_\ell}{R}(2\ell+1) \underbrace{\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta)}_{= \frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$= \frac{\sigma_0}{\varepsilon_0} \int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) \cos\theta\, d(\cos\theta)$$

formula:

$$\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

or

$$\int_{x=-1}^{x=+1} P_{\ell'}(x) P_\ell(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell'\ell}$$

⇒ the sum dissappears!

$$\Rightarrow \frac{C_{\ell'}}{R}\cancel{(2\ell'+1)} \frac{2}{\cancel{2\ell'+1}} = \frac{\sigma_0}{\varepsilon_0} \underbrace{\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) \overbrace{P_1(\cos\theta)}^{\cos\theta} d(\cos\theta)}_{\frac{2}{2\ell'+1}\delta_{\ell'1} = \frac{2}{2(1)+1} = \frac{2}{3}}$$

$\Rightarrow \ell' = 1$

$\Rightarrow$ for $\ell' \neq 1$, $C_{\ell'} = 0$

↳ the entire sum dissappears! [does not always happen] for $\ell' = 1$, $\quad C_1 \frac{\cancel{2}}{R} = \frac{\sigma_0}{\varepsilon_0} \frac{\cancel{2}}{3}$

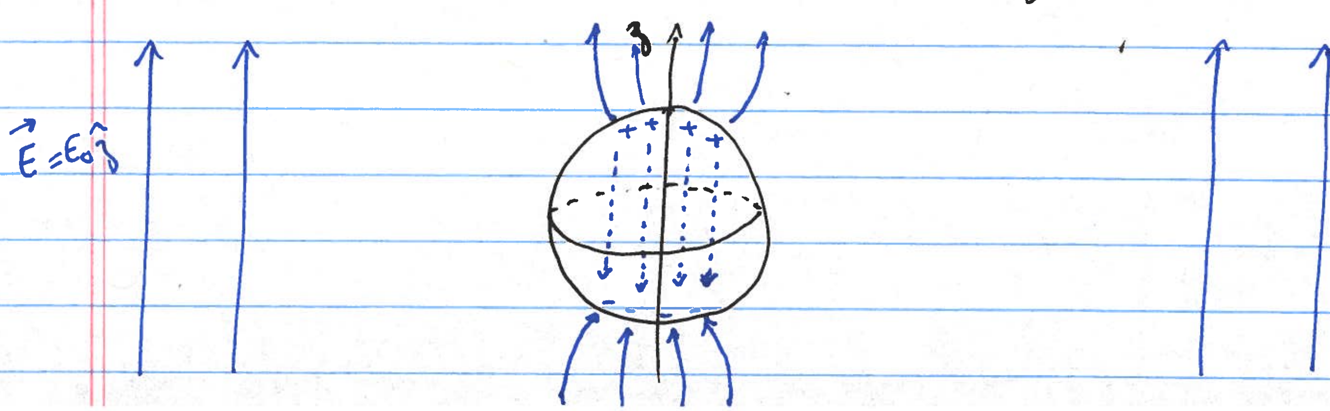
$$\Rightarrow \boxed{C_1 = \frac{\sigma_0}{\varepsilon_0} \frac{R}{3}}$$

$P_1(\cos\theta)$

$$\Rightarrow \begin{cases} V_{in} = \dfrac{\sigma_0}{\varepsilon_0} \dfrac{\cancel{R}}{3} \left(\dfrac{r}{\cancel{R}}\right) \cos\theta \\[2ex] V_{out} = \dfrac{\sigma_0}{\varepsilon_0} \dfrac{R}{3} \left(\dfrac{R}{r}\right)^2 \cos\theta \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} V_{in} = \dfrac{\sigma_0}{3\varepsilon_0} r\cos\theta & (r \leq R) \\[2ex] V_{out} = \dfrac{\sigma_0 R^3}{3\varepsilon_0} \dfrac{\cos\theta}{r^2} & (r \geq R) \end{cases}}$$

Example 2: Solid metal sphere (conducting) in an otherwise uniform electric field $\vec{E} = E_0 \hat{z}$. Calculate $V(r,\theta)$.



Q1: what is the potential at $|\vec{r}| \to +\infty$?
A1: Not zero. Actually $V(r \to \infty) \to \pm\infty, 0$!!!
↳ unusual.

Q2: what is the potential inside the metal sphere?
A2: It must be an equipotential!
↳ $V_{\substack{\text{inside}\\\text{sphere}}}(r,\theta) = cst$.

For convenience, we pick $V_{sphere} = 0$ and put it at the origin.
↑ inside + surface

Q3: what is the potential far from the sphere?
A3: $\vec{E}_{far} = E_0 \hat{z} \implies V = -E_0 z + cst$
$[\vec{E} = -\nabla V]$
↑ $cst = 0$ by symmetry for $V_{sphere} = 0$

$\implies$ Boundary Conditions: (i) $V = 0$ for $r \leq R$
(ii) $V \to -E_0 z$ for $r \gg R$
$= -E_0 r \cos\theta$

general form of solution (outside, for $r \geqslant R$)

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left[a_\ell \left(\frac{r}{R}\right)^\ell + b_\ell \left(\frac{R}{r}\right)^{\ell+1} \right] P_\ell(\cos\theta)$$

boundary condition (i)
if $V(r=R,\theta)=0$, then $a_\ell \cancel{\left(\frac{R}{R}\right)^\ell}^{1} + b_\ell \cancel{\left(\frac{R}{R}\right)^{\ell+1}}^{1} = 0$

$$\implies a_\ell = -b_\ell$$

Thus $$\underset{\text{outside}}{V(r,\theta)} = \sum_{\ell=0}^{\infty} a_\ell \left[\left(\frac{r}{R}\right)^{\ell} - \left(\frac{R}{r}\right)^{\ell+1} \right] P_\ell(\cos\theta)$$

<u>boundary condition (ii)</u>

for $r \gg R$, then $$V(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell \left[\left(\frac{r}{R}\right)^{\ell} - \left(\frac{R}{r}\right)^{\ell+1} \right] P_\ell(\cos\theta)$$

$$= -E_0 r \cos\theta$$

individual terms do not contribute at $r \gg R$

Apply "Fourier trick": $$\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

or just note that $\cos\theta = P_\ell(\cos\theta)$

↳ only $\ell=1$ term is present in sum: $\boxed{a_{\ell\neq 1} = 0}$

$$\Rightarrow a_{\ell=1}\left(\frac{r}{R}\right)^{\ell=1} P_{\ell=1}(\cos\theta) = -E_0 r \cos\theta$$

$$\Leftrightarrow a_1 \frac{\cancel{r}}{R} \cancel{\cos\theta} = -E_0 \cancel{r} \cancel{\cos\theta} \Leftrightarrow \boxed{a_1 = -E_0 R}$$

$$V_{\text{outside}}(r,\theta) = -E_0 R \left[\left(\frac{r}{R}\right) - \left(\frac{R}{r}\right)^2 \right] \cos\theta$$

$$= -E_0 r \cos\theta + E_0 \frac{R^3}{r^2} \cos\theta$$

$$= V_{ext} + E_0 \frac{R^3}{r^2}\cos\theta$$

Q: What's the surface $\underbrace{\text{charge}}_{\text{induced charge}}$ distribution on the sphere?

A: $$-\frac{\sigma(\theta)}{\epsilon_0} = \left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} - \underbrace{\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R}}_{=0} = \underbrace{(-E_{out}) - (-E_{in})}_{-\Delta E}$$

$$= -E_0 \cos\theta + E_0 R^3 \cos\theta \, (-2)\left.\frac{1}{r^3}\right|_{r=R}$$

$$= -E_0 \cos\theta \, (1+2)$$

$$= -3 E_0 \cos\theta \implies \boxed{\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta}$$

$$\hookrightarrow \begin{cases} \sigma(\theta < \pi/2) \text{ is positive} \\ \sigma(\theta > \pi/2) \text{ is negative} \\ \sigma(\theta = \pi/2) = 0 \end{cases}$$