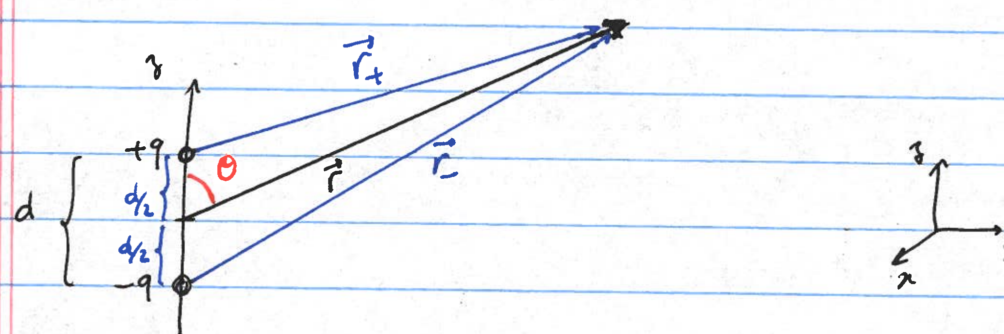
Wednesday, March 24, 2021

Multipole Expansion & Dipole moments

Objective: Describe the electric field at large distances.

Example: Consider a ~~dipole~~ standard dipole



$$\vec{r}_\pm = \mp \frac{d}{2}\hat{z} + \vec{r} \Rightarrow \vec{r}_\pm^{\,2} = r^2 + \left(\frac{d}{2}\right)^2 \mp dr\cos\theta$$

$$\Rightarrow r_\pm = r\sqrt{1 + \left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta}$$

$$\Rightarrow \frac{1}{r_\pm} = \frac{1}{r}\frac{1}{\sqrt{1 + \left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta}}$$

$$\simeq \frac{1}{r}\frac{1}{1 + \frac{1}{2}\left[\underbrace{\left(\frac{d}{2r}\right)^2}_{\text{very small} \to \text{neglect}} \mp \frac{d}{r}\cos\theta\right] \underbrace{- \frac{3}{8}\left[\left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta\right]^2 + \cdots}_{\text{very small} \to \text{neglect}}}$$

$$\simeq \frac{1}{r}\frac{1}{1 \mp \frac{d}{r}\cos\theta}$$

$$\Rightarrow \frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \simeq \frac{1}{r} \pm \frac{d}{2r^2}\cos\theta$$

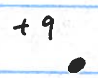
The potential is then

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} + \frac{(-q)}{r_-}\right) = \frac{q}{4\pi\varepsilon_0}\left\{\cancel{\frac{1}{r}} + \frac{d}{2r^2}\cos\theta - \left[\cancel{\frac{1}{r}} - \frac{d}{2r^2}\cos\theta\right]\right\}$$

$$\Rightarrow \boxed{V(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}}$$ (to lowest order in $\frac{d}{r}$)

<u>Long distance behavior:</u>

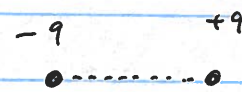
electrostatics



monopole

$V \sim 1/r$

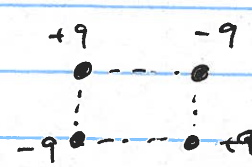
$|\vec{E}| \sim \frac{1}{r^2}$



dipole

$V \sim 1/r^2$

$|\vec{E}| \sim \frac{1}{r^3}$



quadrupole

$V \sim 1/r^3$

$|\vec{E}| \sim 1/r^4$

<u>note:</u> <u>Bare charges</u> are not that common in nature (since they attract an opposing charge to cancel out)

<u>Electric Dipoles</u> are very common (~~th~~ typically electrically neutral)

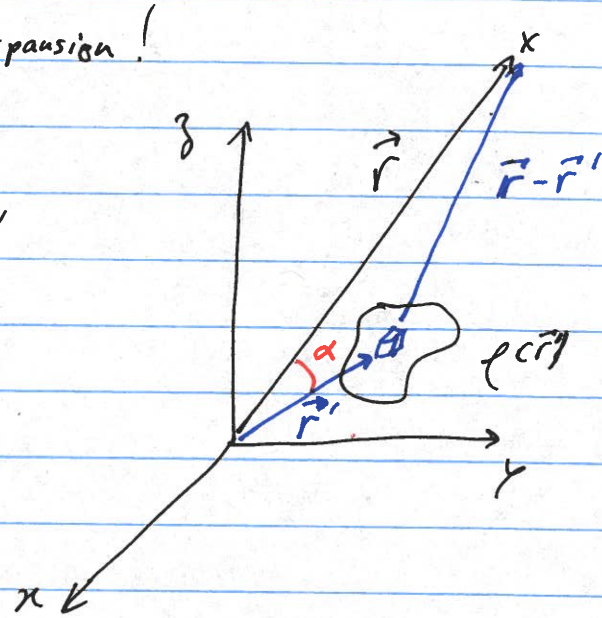
Multipole expansion

Consider a charge distribution $\rho(\vec{r})$.

Q: what is its electric field at large distances (far away)?

A: Use a multipole expansion!

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}$$



(law of cosines) $$|\vec{r}-\vec{r}\,'| = \sqrt{(\vec{r}-\vec{r}\,')^2} = \sqrt{\vec{r}^{\,2} + \vec{r}\,'^2 - 2\,\vec{r}\cdot\vec{r}\,'}$$
$$= \sqrt{r^2 + r'^2 - 2rr'\cos\alpha}$$
$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha}$$

Thus $$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\,\underbrace{\frac{1}{\left[1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha\right]^{1/2}}}_{\frac{1}{\sqrt{1+\varepsilon}} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots}$$

$\left(\text{for } \varepsilon = \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha \ll 1\right)$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\left\{1 - \frac{1}{2}\left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right] + \frac{3}{8}\left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]^2 - \frac{5}{16}\left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]^3 + \dots\right\}$$

$$= \frac{1}{r}\left\{\underbrace{1}_{P_0(\cos\alpha)} + \left(\frac{r'}{r}\right)\underbrace{\cos\alpha}_{P_1(\cos\alpha)} + \left(\frac{r'}{r}\right)^2 \underbrace{\frac{3\cos^2\alpha - 1}{2}}_{P_2(\cos\alpha)} + \left(\frac{r'}{r}\right)^3 \underbrace{\frac{5\cos^3\alpha - 3\cos\alpha}{2}}_{P_3(\cos\alpha)} + \dots\right\}$$

infer, intuit, guess

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$\Rightarrow \boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)} \quad \text{for } r' \ll r$$

$(r' < r)$ is sufficient

Aside: (no proof)

$$P_\ell(\cos\gamma) = P_\ell(\hat{r}\cdot\hat{r}') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y_\ell^{m*}(\theta',\phi')\,Y_\ell^{m}(\theta,\phi)$$

for $\vec{r} = (r, \theta, \phi)$
$\hat{r} = (r', \theta', \phi')$

["addition theorem" for spherical harmonics]

So
$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{4\pi}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{1}{2\ell+1}\left(\frac{r'}{r}\right)^\ell Y_\ell^{m*}(\theta',\phi')\,Y_\ell^{m}(\theta,\phi)$$

Thus
$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|} \\ &= \frac{1}{4\pi\varepsilon_0}\Bigg\{\frac{1}{r}\underbrace{\int \rho(\vec{r}\,')\,d^3r'}_{\text{monopole term}} + \frac{1}{r^2}\underbrace{\int r'\rho(\vec{r}\,')\cos\alpha\, d^3r'}_{\text{dipole term}} \\ &\qquad + \underbrace{\frac{1}{r^3}\int r'^2 \rho(r')\left[\frac{3\cos^2\alpha - 1}{2}\right] d^3r'}_{\text{quadrupole term}} + \cdots\Bigg\}\end{aligned}}$$

Multipole expansion of potential (for $r > r'$)

<u>note:</u> $\int \rho(\vec{r}\,')\,d^3r' = Q$ = total charge

The electric dipole

If the total charge is zero (i.e. $Q = 0$), then to lowest order the field/potential is dipole-like:

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\rho(\vec{r}\,')\underbrace{\cos\alpha}_{\hat{r}\cdot\hat{r}'}\,d^3r'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot}{r^2}\underbrace{\int \rho(\vec{r}\,')\,\vec{r}\,'\,d^3r'}_{\text{electric dipole moment}}$$

definition: $\boxed{\text{electric dipole moment} = \vec{p} = \int\limits_{\substack{\text{all space} \\ \text{or} \\ \text{volume of } \rho(\vec{r}\,')}} d^3r' \, \rho(\vec{r}\,') \, \vec{r}\,'}$

$$\Rightarrow \boxed{V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}}$$

Q: Is $\vec{p}$ ~~indep~~ an intrinsic quantity independent of the coordinate system?

Consider the electric dipole moment: $\vec{p} = \int d^3r' \, \rho(\vec{r}\,') \, \vec{r}\,'$

Now shift $\rho(\vec{r})$ by $\vec{d}$, i.e. $\rho_d(\vec{r}) = \rho(\vec{r} - \vec{d})$

$$\Rightarrow \vec{p}_d = \int d^3r' \, \underbrace{\rho(\vec{r}\,' - \vec{d})}_{\rho_d(\vec{r}\,')} \, \vec{r}\,'$$

substitution:
$\vec{r}\,'' = \vec{r}\,' - \vec{d}$
$d^3r'' = d^3r'$

$$= \int d^3r'' \, \rho(\vec{r}\,'') (\vec{r}\,'' + \vec{d})$$

$$= \underbrace{\int d^3r' \, \rho(\vec{r}\,'') \, \vec{r}\,''}_{\vec{p}} + \vec{d} \underbrace{\int d^3r'' \, \rho(\vec{r}\,'')}_{Q}$$

Thus $\vec{p}_d = \vec{p} + Q\vec{d}$

A: $\Rightarrow$ The dipole moment depends on the choice of origin!

$\Rightarrow$ For a neutral system, the dipole moment is independent of origin.
(i.e. it is coordinate free)

Example: Dipole of N point charges

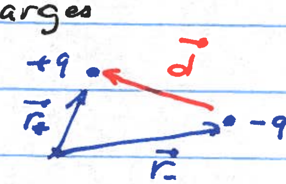
examples:
- atom: nucleus $+ e^-$'s
- molecule: H_2O in a weak E-field

$$\rho(\vec{r}) = \sum_{i=1}^{N} q_i \, \delta(\vec{r} - \vec{r}_i)$$

thus $\vec{p} = \int d^3r' \sum_{i=1}^{N} q_i \, \delta(\vec{r}\,' - \vec{r}_i) \, \vec{r}\,'$

$$\Rightarrow \boxed{\vec{p} = \sum_{i=1}^{N} q_i \, \vec{r}_i}$$

Special case: 2 equal & opposite point charges



$$\vec{p} = q\vec{r}_+ - q\vec{r}_-$$
$$= q(\vec{r}_+ - \vec{r}_-)$$
$$= q\vec{d} \quad \Rightarrow \quad \vec{p} = q\vec{d}$$

points from $\underline{-q}$ to $\underline{+q}$

Electric Field of a dipole: $\vec{E} = -\vec{\nabla} V = -\vec{\nabla}\left(\frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}\right)$

some algebra

$$\Rightarrow \boxed{\vec{E}_{dipole} = \frac{1}{4\pi\epsilon} \frac{3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}}{r^3}}$$

coordinate free form → useful

some algebra

$$\Rightarrow \vec{E}_{dipole} = \frac{p}{4\pi\epsilon_0}\left[2\cos\theta \, \hat{r} + \sin\theta \, \hat{\theta}\right]$$

[for $\vec{p} = p\hat{z}$]