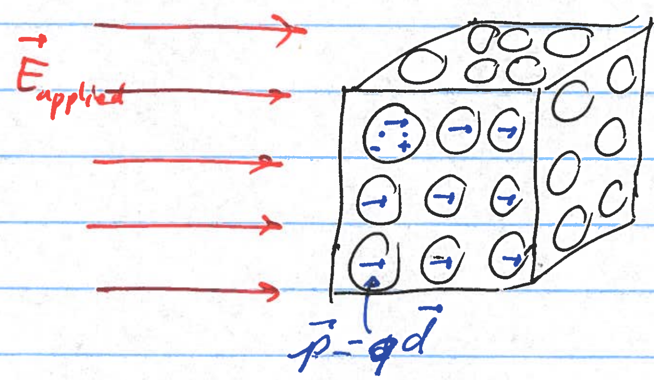
Wednesday, March 31, 2021

Summary from last time

- We can model matter as a collection of little dipoles with $\vec{p} \overset{?}{=} \alpha \vec{E}_{applied}$



- Definition: $\vec{P} \equiv$ dipole moment per unit volume i.e. " $\frac{N q\vec{d}}{m^3}$ "

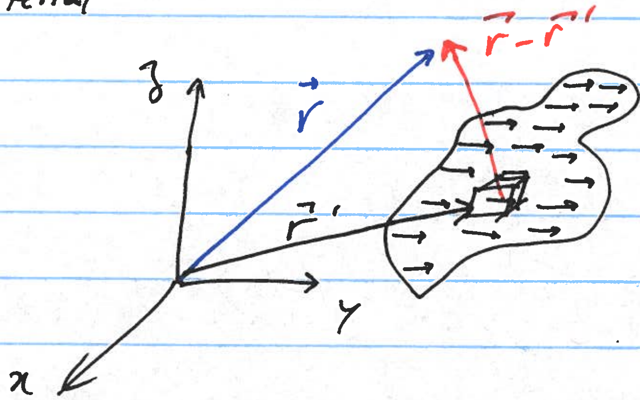
- Charges inside a material may not exactly cancel inside a material subject to an applied E-field.

$$\hookrightarrow \begin{cases} \text{Volume bound charge: } \rho_b(\vec{r}) \\ \text{Surface bound charge: } \sigma_b(\vec{r}) \end{cases}$$

Q: What is the relationship between $\vec{P}$, ρ_b, and σ_b

A: Let's derive it!

Consider the potential $V(\vec{r})$ produced by a polarized material



For a single dipole $\vec{p}$: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$

For a continuous distribution of $\vec{p}$'s: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\vec{P}(\vec{r}\,')\cdot\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2} d^3r'$
(i.e. our material)

However, we note that $\vec{\nabla}_{r'}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = \frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}$

a few lines of algebra

Thus, $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \underbrace{\vec{P}(\vec{r}\,')\cdot\vec{\nabla}_{r'}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)}_{\vec{\nabla}\cdot\left(\vec{P}\frac{1}{|\vec{r}-\vec{r}\,'|}\right) - \frac{1}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}\cdot\vec{P}} d^3r'$

$$= \frac{1}{4\pi\varepsilon_0}\left\{ \underbrace{\int_V \vec{\nabla}\cdot\left(\vec{P}\frac{1}{|\vec{r}-\vec{r}\,'|}\right) d^3r'}_{\text{divergence theorem}} + \int_V \frac{(-\vec{\nabla}\cdot\vec{P})}{|\vec{r}-\vec{r}\,'|} d^3r' \right\}$$

$$\Rightarrow \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \underbrace{\frac{\vec{P}(\vec{r}\,')\cdot\hat{n}'\,ds'}{|\vec{r}-\vec{r}\,'|}} + \frac{1}{4\pi\varepsilon_0}\underbrace{\int_V \frac{\left(-\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r}\,')\right)}{|\vec{r}-\vec{r}\,'|}\,d^3r'}$$

Surface charge distribution

Standard form for the potential produced by a charge distribution

Thus we infer (or define):

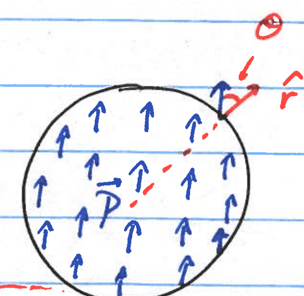
$$\rho_b(\vec{r}) \equiv -\vec{\nabla}\cdot\vec{P}(\vec{r})$$

$$\sigma_b(\vec{r}) \equiv \vec{P}(\vec{r})\cdot\hat{n}$$

where $\hat{n} \equiv$ normal unit vector pointing out of surface

<u>Example</u>: Calculate the potential of a uniformly polarized dielectric sphere (radius $= R$) with polarization $\vec{P} =$ cst.

<u>Solution</u>: pick $\vec{P} = P\hat{z}$



$$\vec{\nabla}\cdot\vec{P} = 0 \quad \Rightarrow \quad \boxed{\rho_b(\vec{r}) = 0}$$

$$\vec{P}\cdot\hat{n} = P\hat{z}\cdot\hat{r} = P\cos\theta \quad \Rightarrow \quad \boxed{\sigma_b(\vec{r}) = P\cos\theta}$$

Thus
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma_b(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^2r' = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{P\cos\theta'\,R^2\sin\theta'\,d\theta'\,d\phi'}{|\vec{r}-\vec{r}\,'|}$$

<u>or</u> use separation of variables!

from lecture 14 (Monday, March 22), we obtained

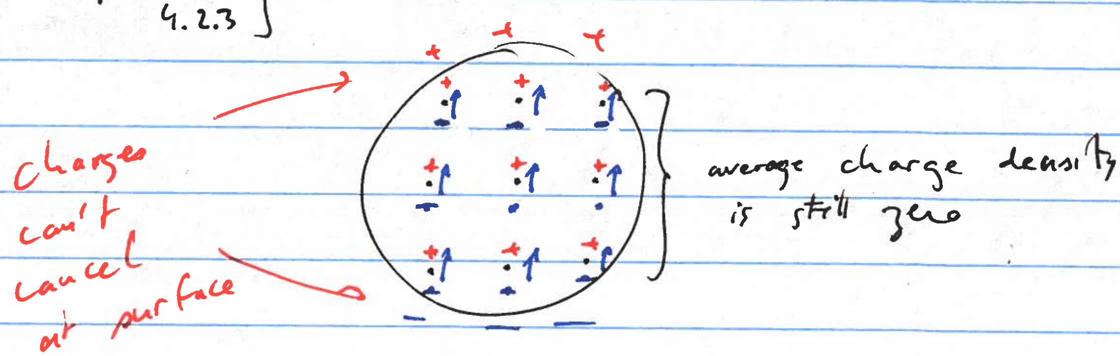
$$V_{in} = \frac{\overset{P}{\sigma_0}}{3\varepsilon_0} \underbrace{r\cos\theta}_{z} \qquad \text{for } r \leq R$$

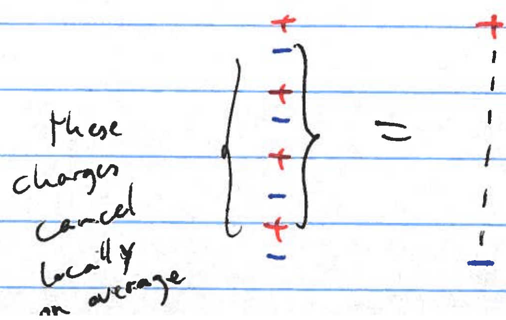
$$V_{out} = \frac{\overset{P}{\sigma_0} R^3}{3\varepsilon_0} \frac{\cos\theta}{r^2} \qquad \text{for } r \geq R$$

$$= \frac{1}{4\pi\varepsilon_0} \underbrace{\underbrace{\frac{4\pi R^3}{3}}_{\text{volume}} \underbrace{P}_{\text{polarization per volume}}}_{\text{total dipole moment } \mathbb{P}\hat{z}} \cos\theta \frac{1}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\vec{\mathbb{P}} \cdot \hat{r}}{r^2} \Rightarrow \text{it's a perfect dipole field/potential.}$$

Q: Why is ~~also~~ all the bound charge on the surface?
[see chpt. 4.2.2
4.2.3]





Gauss's Law and Electric Displacement $\vec{D}$

Gauss's law always applies, thus inside a dielectric we have

controlled by experiment

$$\vec{\nabla}\cdot \underbrace{\vec{E}_{total}(\vec{r})}_{\vec{E}} = \frac{\rho_{total}(\vec{r})}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\rho_{free} + \rho_b\right)$$

from material $-\vec{\nabla}\cdot\vec{P}$

$$\Rightarrow \quad \varepsilon_0\, \vec{\nabla}\cdot\vec{E} = \rho_{free} - \vec{\nabla}\cdot\vec{P}$$

$$\Leftrightarrow \quad \vec{\nabla}\cdot\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_{free}$$

$\vec{D}(\vec{r})$ vector field

we define the electric displacement $\vec{D}$ as $\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}}$

$\vec{D}$ obeys "Gauss's law": $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_{free}}$

$$\Leftrightarrow \quad \boxed{\oint_S \vec{D}\cdot d\vec{s} = Q_{free,\,enclosed}}$$

$\Rightarrow$ "Electrostatics as usual", but with $\vec{D}$ instead $\vec{E}$, $\varepsilon_0 \to 1$, and ρ_f instead ρ.

⚠ $\vec{\nabla}\times\vec{D} = \varepsilon_0 \underbrace{\vec{\nabla}\times\vec{E}}_{=0} + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P} \neq 0 \Rightarrow$ not quite electrostatics as usual.

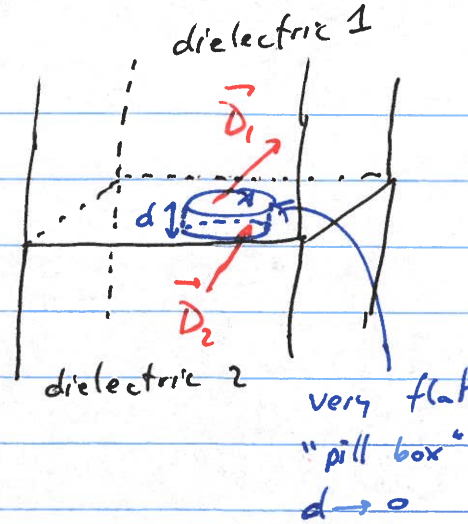
especially at a surface

there is no displacement potential!

note: If you only use "Gauss's law" + symmetry then ~~[illegible]~~ getting $\vec{D}$ is straightforward.

Boundary Conditions

At a dielectric interface



$$\oint_S \vec{D}\cdot d\vec{s} = Q_{free,enclosed}$$

$$\Rightarrow \quad A\vec{D}_1\cdot\hat{n}_1 + A\vec{D}_2\cdot\underbrace{\hat{n}_2}_{-\hat{n}_1} = \sigma_{free}\,A$$

$$\Rightarrow \quad \boxed{\left(\vec{D}_1 - \vec{D}_2\right)_\perp = \sigma_{free}}$$

note:

$$\left(\vec{E}_1 - \vec{E}_2\right)_\perp = \frac{\sigma_f + \sigma_b}{\varepsilon_0}$$

Also, at a charged surface $\hat{n}\times\Delta\vec{E} = 0$

$$\Leftrightarrow \quad \hat{n}_1\times\left(\vec{E}_1 - \vec{E}_2\right) = 0$$

(equivalent to $\Delta\vec{E}_{/\!/} = 0$)

$$\left(\vec{E}_1 - \vec{E}_2\right)_{/\!/} = 0$$

then $\hat{n}_1\times\left(\vec{D}_1 - \vec{D}_2\right) = \underbrace{\hat{n}_1\times\left[\varepsilon_0\left(\vec{E}_1 - \vec{E}_2\right)\right.}_{=0} \left. + \vec{P}_1 - \vec{P}_2\right]$

$$= \hat{n}_1\times\left(\vec{P}_1 - \vec{P}_2\right)$$

$$\Rightarrow \quad \boxed{\left(\vec{D}_1 - \vec{D}_2\right)_{/\!/} = \left(\vec{P}_1 - \vec{P}_2\right)_{/\!/}}$$