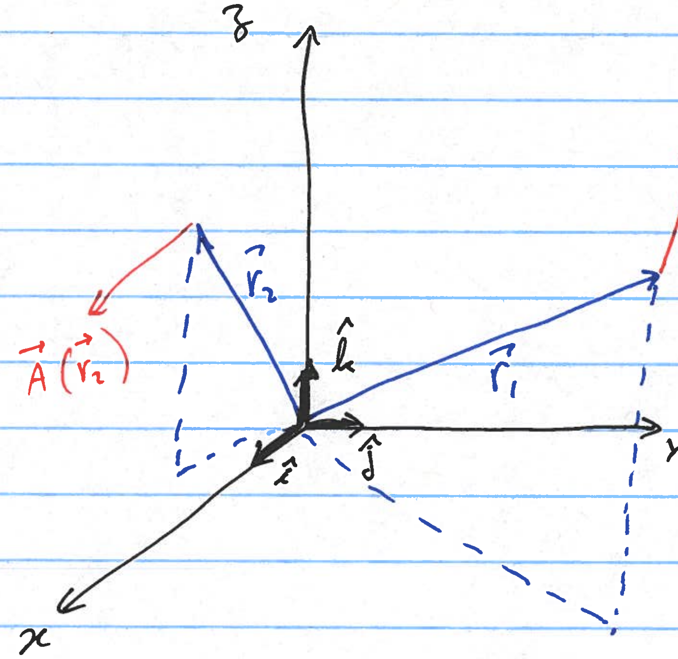
Wednesday, January 25, 2023

Basic Review of Vector Calculus

I Vector Fields (cartesian coordinates)

Definition (physics): Every point in space (3D) has a vector associated with it.

(i.e. Field) ← typically abstract, non-spatial, 3D



$$\vec{A}(\vec{r}_1) = \left(A_x(\vec{r}_1), A_y(\vec{r}_1), A_z(\vec{r}_1)\right)$$

$$= \left(A_x(x,y,z), A_y(x,y,z), A_z(x,y,z)\right)$$

$$= A_x(\vec{r}_1)\,\underset{\hat{x}}{\hat{i}} + A_y(\vec{r}_1)\,\underset{\hat{y}}{\hat{j}} + A_z(\vec{r}_1)\,\underset{\hat{z}}{\hat{k}}$$

Generally / typically, the vector field components (A_x, A_y, A_z) are continuous and differentiable functions of the Cartesian coordinates $\vec{r} = (x, y, z)$ [and time t]

not often in this course!

II Vector Products of two vectors $\vec{A}$ & $\vec{B}$
or vector fields

A. Scalar product ("dot" product) = number

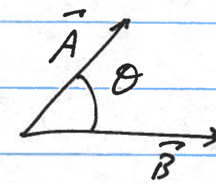
$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z = \vec{B}\cdot\vec{A}$$

(commutative)

$$= |\vec{A}||\vec{B}| \cos\theta_{AB}$$

angle between $\vec{A}$ & $\vec{B}$

$\vec{A}(\vec{r}_1)\cdot\vec{B}(\vec{r}_1)$



⚠ note: If you find $\vec{A}(\vec{r}_1)\cdot\vec{B}(\vec{r}_2)$, then there is a mistake!

not local!

Vector magnitude (i.e. "length"): $A = |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$

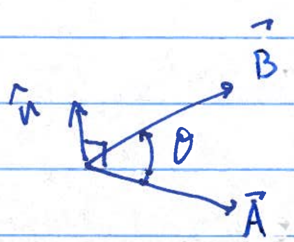
B. Vector product ("cross" product) = vector

$$\vec{A}\times\vec{B} = \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}_\times = -\vec{B}\times\vec{A}$$

$$= \left(A_y B_z - A_z B_y,\ A_z B_x - A_x B_z,\ A_x B_y - A_y B_x\right)$$

$$= |\vec{A}||\vec{B}| \sin\theta_{AB}\ \hat{n}$$

unit vector perpendicular to $\vec{A}-\vec{B}$ plane, in direction given by right hand rule



III Vector field "derivatives"

single component vector field

A_ Gradient

The gradient of a scalar field $f(x,y,z)$ is the vector:

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$= \partial_x f \, \hat{x} + \partial_y f \, \hat{y} + \partial_z f \, \hat{z}$$

Direction of $\vec{\nabla} f$: $\vec{\nabla} f$ points in the direction of maximum increase of the function $f(x,y,z)$

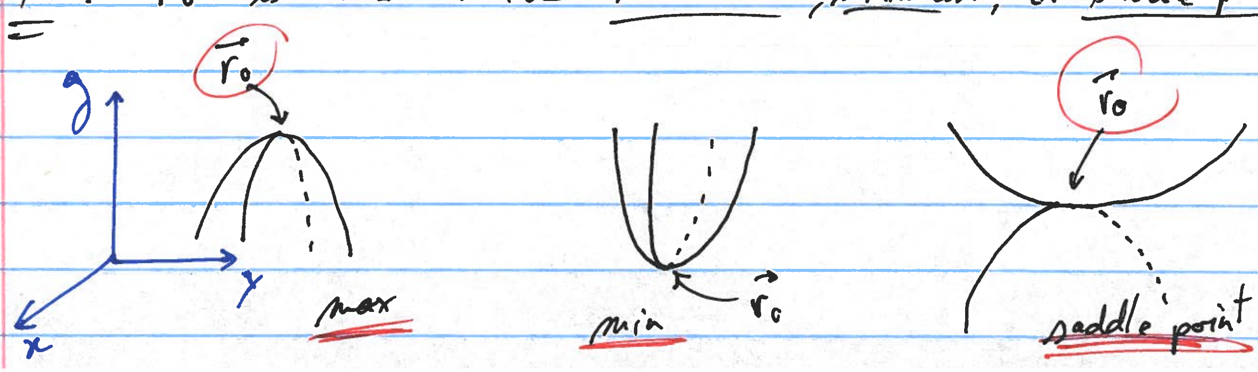
Magnitude of $\vec{\nabla} f$: $|\vec{\nabla} f|$ gives the slope along the direction of maximum increase.

Del operator $\vec{\nabla}$: (vector operator) $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

Q: If $\vec{\nabla} g(x,y,z) = \vec{0} = (0,0,0)$ for $\vec{r}_0 = (x_0, y_0, z_0)$, then what can we say about $\vec{r}_0$?

A: $\vec{r}_0$ is either a local maximum, minimum, or saddle point.

in 2D



Example: Calculate $\vec{\nabla}\frac{1}{r}$ $\left[\text{recall: } r = |\vec{r}| = \sqrt{x^2+y^2+z^2}\right]$

$$\vec{\nabla}\frac{1}{r} = (\partial_x, \partial_y, \partial_z)\underbrace{\frac{1}{\sqrt{x^2+y^2+z^2}}}_{=(x^2+y^2+z^2)^{-1/2}}$$

$$= \left(\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}}, \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left(-\frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

note: $(x^2+y^2+z^2)^{3/2} = (r^2)^{3/2} = r^3 = r^2 r$

$$= \underbrace{\left(\frac{-x}{r^2 r}, \frac{-y}{r^2 r}, \frac{-z}{r^2 r}\right)}$$ but $\vec{r} = (x, y, z)$

$$= -\frac{\vec{r}}{r^2 r} = -\frac{\hat{r}}{r^2}$$

($\frac{\vec{r}}{r} = \hat{r}$)

$$\Rightarrow \boxed{\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}}$$

B- Divergence

The divergence of a vector field $\vec{A}(\vec{r})$ is a scalar:

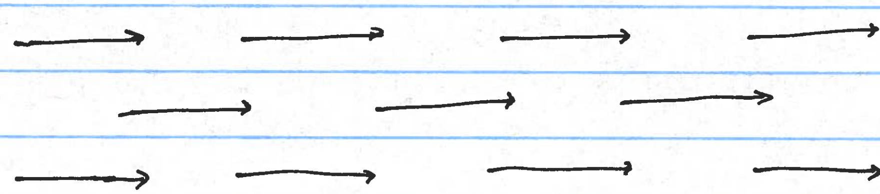
Stopped here

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

Example 1: $\vec{A}(\vec{r}) = v_0\,\hat{x}$

$= (v_0, 0, 0)$, i.e. "constant flow" if you think of the vector field as fluid velocity map.

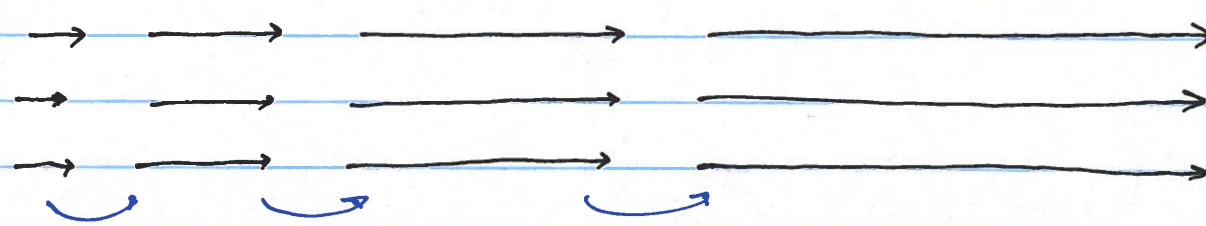
$$\vec{\nabla}\cdot\vec{A} = \cancel{\frac{\partial}{\partial x} v_0}^{=0} + \underbrace{\frac{\partial}{\partial y}0}_{=0} + \underbrace{\frac{\partial}{\partial z}0}_{=0} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{A} = 0}$$



Example 2: $\vec{A}(\vec{r}) = (ax, 0, 0)$

i.e. "increasing flow" (or "accelerating" flow)

$$\vec{\nabla}\cdot\vec{A} = \cancel{\frac{\partial}{\partial x} ax}^{a} + \underbrace{\frac{\partial}{\partial y}0}_{=0} + \underbrace{\frac{\partial}{\partial z}0}_{=0} = a$$



If fluid flow increases (velocity increases) ~~then~~ in a constant density medium, then fluid must have been created.