Monday, January 30, 2023 #1

III Vector Field Derivatives (continued)

B. Divergence

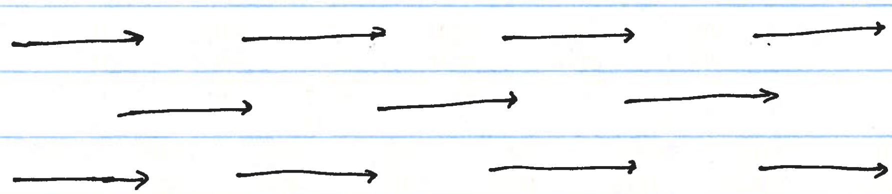
The divergence of a vector field $\vec{A}(\vec{r})$ is a <u>scalar</u>:

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

<u>Example 1:</u> $\vec{A}(\vec{r}) = v_0\,\hat{x}$

$= (v_0, 0, 0)$, i.e. "constant flow" if you think of the vector field as fluid velocity map.

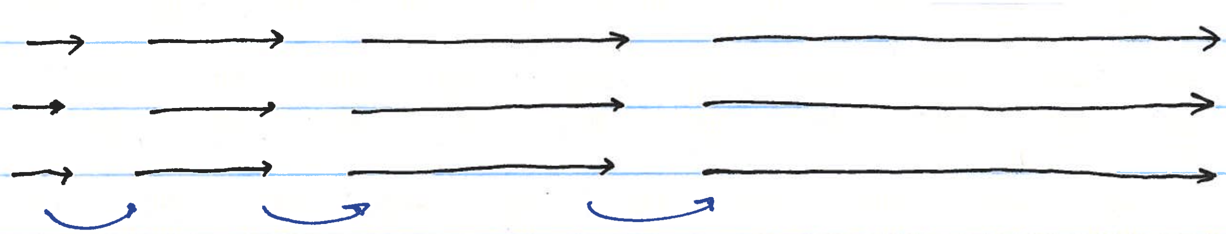
$$\vec{\nabla}\cdot\vec{A} = \overbrace{\cancel{\frac{\partial}{\partial x} v_0}}^{=0} + \underbrace{\frac{\partial}{\partial y}0}_{=0} + \underbrace{\frac{\partial}{\partial z}0}_{=0} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{A} = 0}$$



<u>Example 2:</u> $\vec{A}(\vec{r}) = (ax, 0, 0)$

i.e. "increasing flow" (or "accelerating" flow)

$$\vec{\nabla}\cdot\vec{A} = \overbrace{\cancel{\frac{\partial}{\partial x} ax}}^{a} + \underbrace{\frac{\partial}{\partial y}0}_{=0} + \underbrace{\frac{\partial}{\partial z}0}_{=0} = a$$



If fluid flow increases (velocity increases) ~~then~~ in a <u>constant density medium</u>, then fluid must have been <u>created</u>.

Divergence is positive when "fluid" is created/sourced

Divergence is negative when "fluid" is eliminated/"sunk"

C. Curl

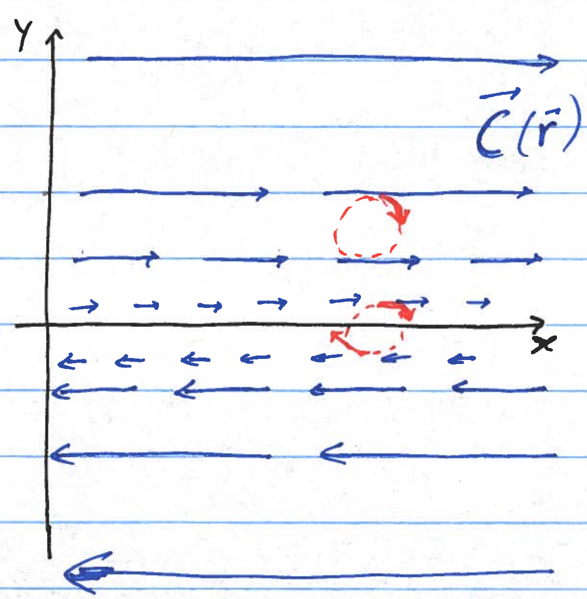
The curl of a vector field $\vec{A}(\vec{r})$ is a vector:

$$\vec{\nabla}\times\vec{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(A_x, A_y, A_z\right)$$

$$= \left(\partial_y A_z - \partial_z A_y,\ \partial_z A_x - \partial_x A_z,\ \partial_x A_y - \partial_y A_x\right)$$

Example 1: Both $\underbrace{\vec{A} = v_0 \hat{x}}_{\vec{\nabla}\times\vec{A}=\vec{0}}$ and $\underbrace{\vec{B} = a x \hat{x}}_{\vec{\nabla}\times\vec{B}=\vec{0}}$ have curl $= 0$.

Example 2: $\vec{C} = c y \hat{x} = (cy, 0, 0)$



$$\vec{\nabla}\times\vec{C} = (0, 0-0, 0-c) = -c\hat{z} \neq \vec{0}$$

The curl is non-zero when the "fluid flow" of the vector field can "rotate" (partially) a finite particle, e.g. "cheerio".

D_ Laplacian

Consider a scalar field $f(\vec{r})$. The Laplacian of f is given by:

$$\nabla^2 f = \Delta f = \underbrace{\vec{\nabla}\cdot\left(\vec{\nabla} f\right)}_{\text{"div-grad"}} = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$$

The Laplacian of a vector field $\vec{A}(\vec{r})$ is given by

$$\nabla^2 \vec{A} = \Delta \vec{A} = \left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{A} = \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z\right)$$

$$\left(\Delta A_x, \Delta A_y, \Delta A_z\right)$$

$$\underbrace{\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)}_{\text{"grad-div"}} \neq \nabla^2 \vec{A}$$ and is not used much on its own.

E_ other second derivatives

$f(\vec{r}) =$ scalar field , $\vec{A}(\vec{r}) =$ vector field

Div-Curl : $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$

some algebra [recall: $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$]

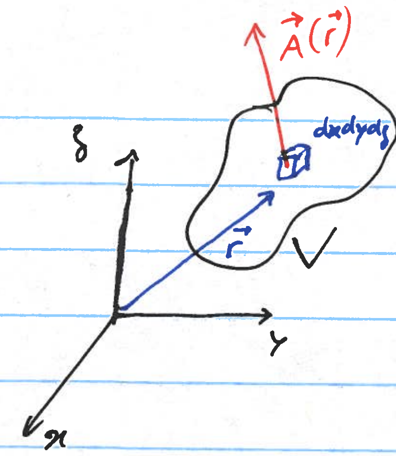
Curl-grad : $\vec{\nabla}\times\left(\vec{\nabla} f\right) = \vec{0}$

some algebra

Curl-Curl : $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A}$

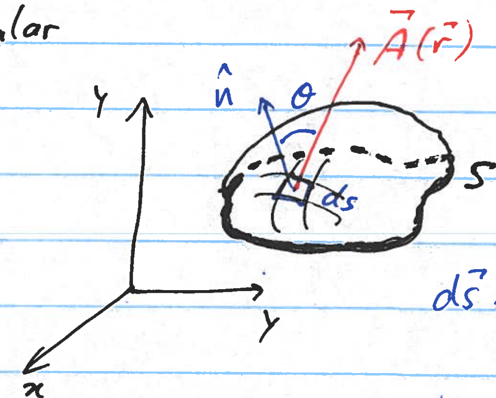
some algebra

IV Integrals



volume: $\displaystyle\int_V \vec{A}(\vec{r})\, \underbrace{dx\,dy\,dz}_{dV = d^3r} = \text{vector}$

Surface: $\displaystyle\int_S \vec{A}(\vec{r}) \cdot d\vec{s} = \text{scalar}$



$d\vec{s} = ds \cdot \hat{n}$

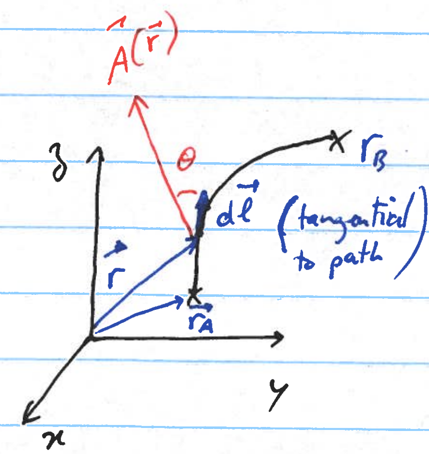
unit vector normal to surface

note: for a closed surface (e.g. balloon, potato skin)

$d\vec{s}$ sticks out by convention.

↳ notation: $\displaystyle\int_S \longrightarrow \oint_S$ S = closed surface

Line: $\displaystyle\int_{\vec{r}_A}^{\vec{r}_B} \vec{A}(\vec{r}) \cdot d\vec{l} = \text{scalar}$ path



↳ closed path: $\displaystyle\oint_{path} \vec{A} \cdot d\vec{l} = \text{scalar}$

V Integral theorems

A. Gradient theorem

Consider a vector field $\vec{A}(\vec{r}) = \vec{\nabla} f(\vec{r})$, then

$$\int_{\substack{\vec{r}_A \\ \text{path } P}}^{\vec{r}_B} \vec{A} \cdot d\vec{\ell} = \boxed{\int_{\vec{r}_A}^{\vec{r}_B} \left(\vec{\nabla} f(\vec{r})\right) \cdot d\vec{\ell} = f(\vec{r}_B) - f(\vec{r}_A)}$$

$f(\vec{r})$ is called the potential of $\vec{A}$

corollary: The integral depends only on the endpoints $\vec{r}_A$ & $\vec{r}_B$, not the path.

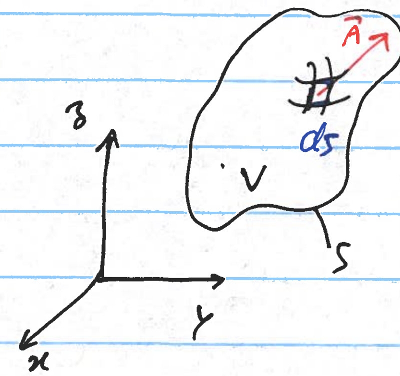
↳ note: $\displaystyle\oint_{\substack{\text{closed path } P \\ (\text{loop})}} \vec{\nabla} f \cdot d\vec{\ell} = 0$

example: In a DC electric circuit, the voltage difference between A & B does not depend on the shape of the wire.

B. Divergence Theorem / Gauss's theorem ["Green's theorem"]

Consider a vector field $\vec{A}(\vec{r})$ and a volume V with surface S, then

$$\boxed{\int_V (\vec{\nabla}\cdot\vec{A})\,dV = \oint_S \vec{A}\cdot d\vec{s}}$$

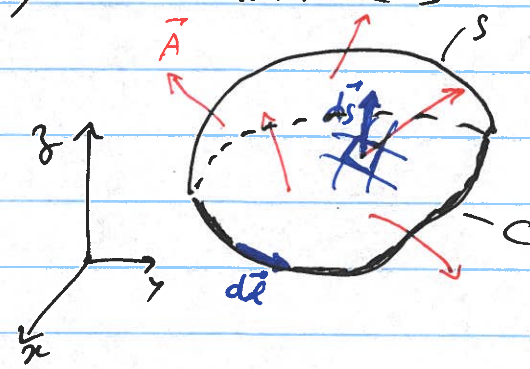


In words:

$$\text{“} \int \text{sources/sinks in } V = \oint \text{flow out/in through } S \text{”}$$

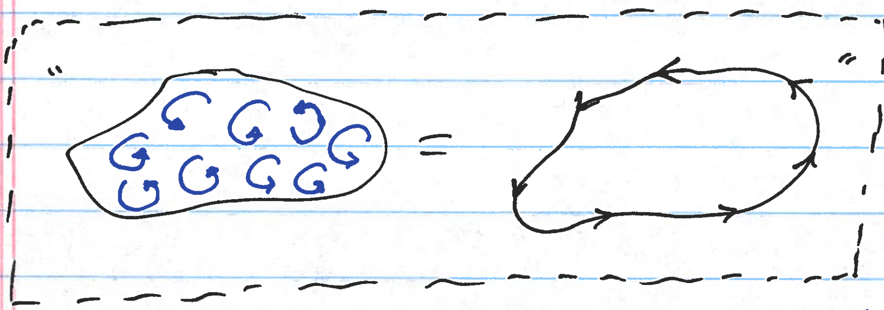
C. Stokes's Theorem (Curl theorem)

Consider a vector field $\vec{A}(\vec{r})$ and a surface S with bounding line C, then



$$\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \oint_C \vec{A}\cdot d\vec{\ell}$$

In pictures:



$\vec{\nabla}\times\vec{A}$ = local circulation (i.e. fluid "rotation") of the field $\vec{A}$

note 1: $\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$ does not depend on the surface S, but only on the boundary C.

note 2: $\oint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = 0$ for a closed surface S.