Wednesday, February 1, 2023

(V) Integral theorems (continued)

D – Integration by parts

$f(\vec{r})$ = scalar field
$\vec{A}(\vec{r})$ = vector field
V = volume with surface S

" $\int_a^b u\,dv$ "

$$\int_V \underbrace{f(\vec{r})\left(\vec{\nabla}\cdot\vec{A}(\vec{r})\right)}\, d^3r$$

from front of book: $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$

$\Leftrightarrow \quad f(\vec{\nabla}\cdot\vec{A}) = \vec{\nabla}\cdot(f\vec{A}) - \vec{A}\cdot(\vec{\nabla}f)$

$$= \underbrace{\int_V \vec{\nabla}\cdot(f\vec{A})\, d^3r}_{\text{Apply Divergence theorem}} - \int_V \vec{A}\cdot(\vec{\nabla}f)\, d^3r$$

$$= \underbrace{\int_S f\vec{A}\cdot d\vec{s}} - \int_V \vec{A}\cdot(\vec{\nabla}f)\, d^3r$$

" $uv\Big|_a^b - \int_a^b v\,du$ "

→ you only need to know $f(\vec{r})$ and $\vec{A}(\vec{r})$ on the surface

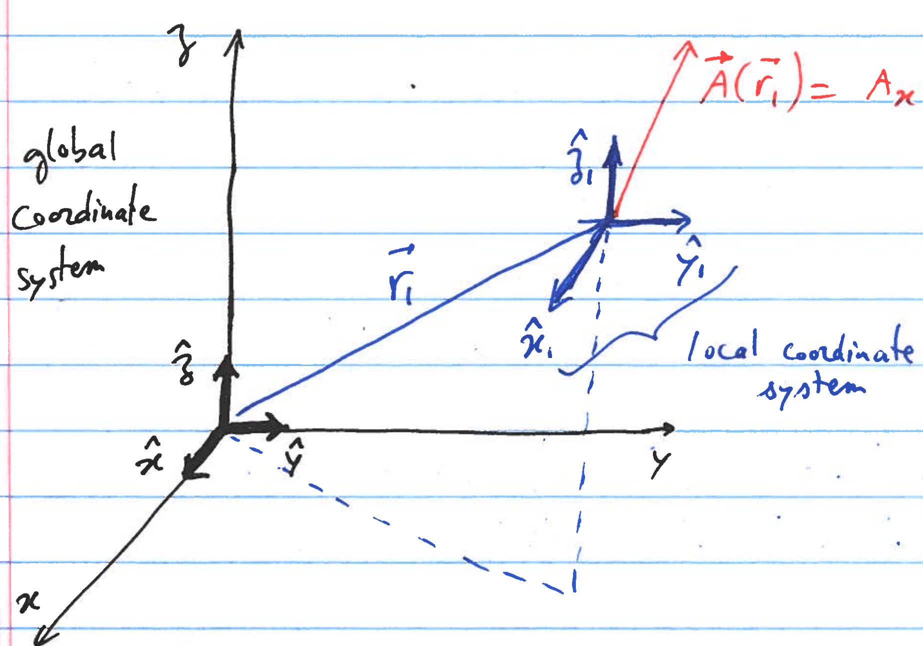
→ in E&M, often fields are zero on a boundary (i.e. boundary condition)

note: The same approach can be used to apply Stokes's theorem to certain surface integrals.

VI Spherical Coordinates (as an example of curvilinear coordinates)

Reminder about Cartesian Coordinates

$\vec{A}(\vec{r}_1) = A_x\,\hat{x}_1 + A_y\,\hat{y}_1 + A_z\,\hat{z}_1$



Important:

$\hat{x}_1 = \hat{x} = \ldots = \hat{x}_n$

$\hat{y}_1 = \hat{y} = \ldots = \hat{y}_n$

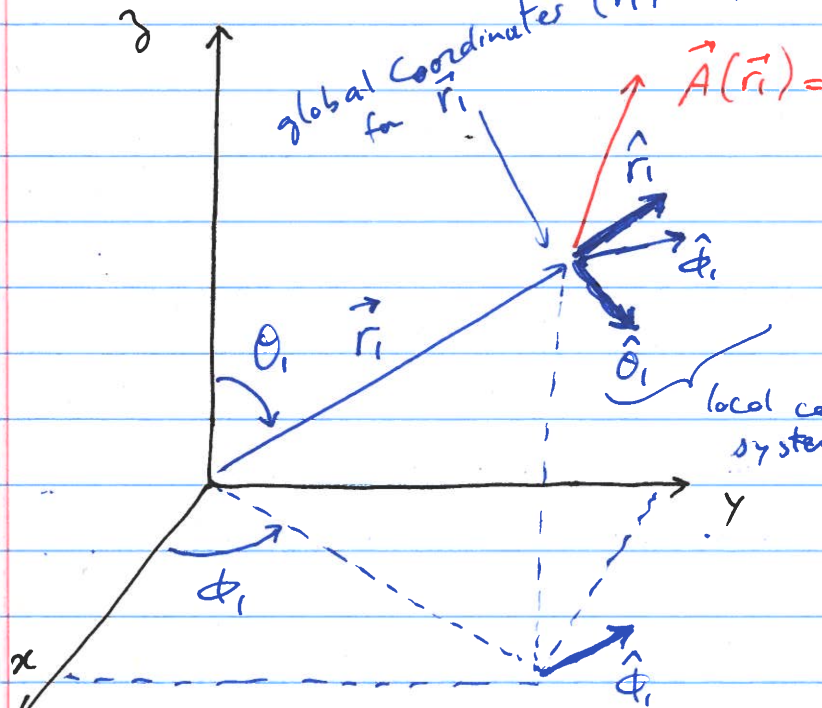
$\hat{z}_1 = \hat{z} = \ldots = \hat{z}_n$

(very convenient)

Spherical Coordinates (global & local)

global coordinates (r_1, θ_1, ϕ_1) for $\vec{r}_1$

$\vec{A}(\vec{r}_1) = A_{r_1}\,\hat{r}_1 + A_{\theta_1}\,\hat{\theta}_1 + A_{\phi_1}\,\hat{\phi}_1$



local coordinate (orthonormal) system

Important:

$\hat{r}_1 \neq \hat{r}_n$

$\hat{\theta}_1 \neq \hat{\theta}_n$

$\hat{\phi}_1 \neq \hat{\phi}_n$

Global coordinate conversion

$$\begin{cases} z = r\cos\theta \\ y = r\sin\theta\sin\phi \\ = r\sin\theta\cos\phi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\dfrac{x^2+y^2}{z}\right) \\ \phi = \arctan\left(\dfrac{Y}{x}\right) \end{cases}$$

local coordinate conversion :

⚠ The local coordinate system unit vectors depend on (θ, ϕ)

$$\hat{r}(\theta,\phi) = \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta}(\theta,\phi) = \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi}(\theta,\phi) = \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

Recommendation (sometimes useful)

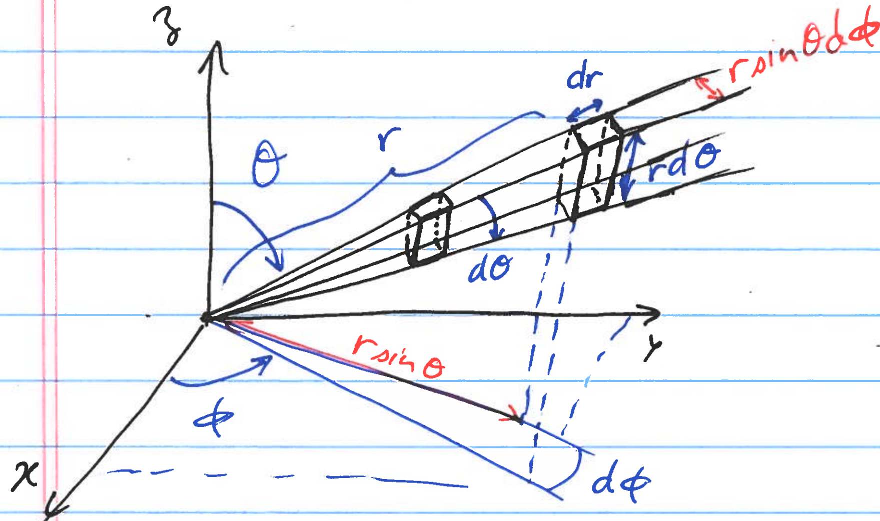
For a complicated problem with spherical symmetry, start in spherical coordinates, but then convert to cartesian coordinates.

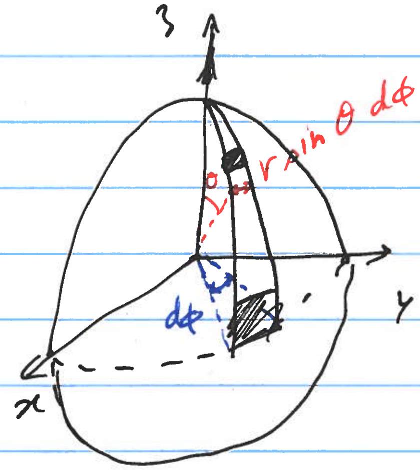
Infinitesimal Volume Integration Element

Cartesian coordinates : $dV = d^3r = dx\,dy\,dz\,(x_1, y_1, z_1) = dx\,dy\,dz\,(x_n, y_n, z_n)$

↳ independent of location.

Spherical coordinates





⚠ The volume of the volume element depends on the coordinates!

$$dV = dr \cdot r d\theta \cdot r \sin\theta \, d\phi$$

$$\Leftrightarrow \boxed{dV = r^2 \sin\theta \, dr \, d\theta \, d\phi}$$

→ replace $dx\,dy\,dz$ with this expression

note: $dx\,dy\,dz \neq r^2 \sin\theta \, dr\, d\theta\, d\phi$

more generally:
$$dV = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}}_{\text{determinant of the Jacobian matrix} \; = r^2 \sin\theta} dr \, d\theta \, d\phi$$

For cylindrical coordinates: $dV = r\,dr\,dz\,d\phi$

Grad, Div, Curl, etc

Gradient:

$$\vec{\nabla} f(\vec{r}) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

lots of algebra

$$= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

depends on location!

ex:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\phi$$

also $\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$

etc...

#6

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right)}_{\frac{1}{r}\frac{\partial^2}{\partial r^2}(rt)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VII Divergence of $\vec{A} = \hat{r}/r^2$

[⚠ plays a central role in E&M]

Divergence in spherical coordinates:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, \cancelto{0}{A_\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial \cancelto{0}{A_\phi}}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\cancelto{0}{\frac{\partial}{\partial r}(1)} = \frac{0}{r^2}$$

$$= 0$$

⚠ wrong !!

"paradox":

$$\int_{V = \text{sphere of radius } R}(\vec{\nabla}\cdot\vec{A})\,dV = \int_{S = \text{surface of sphere}}\vec{A}\cdot d\vec{s} = 4\pi R^2 \frac{1}{r^2}\Big|_{r=R}$$

$$\int \text{"0"}\, dV = 0$$

$$\neq$$

$$= 4\pi$$

does not depend on "R"

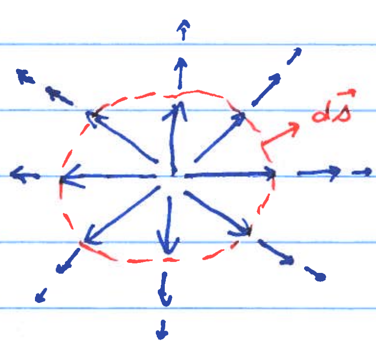
even if $R = \varepsilon$

i.e. "infinitesimal sphere" centered on origin

Resolution:

$$\boxed{\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\,\delta^3(\vec{r})}$$

where $\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$



note: $$\vec{\nabla}\cdot\underbrace{\left(\vec{\nabla}\frac{1}{r}\right)}_{-\hat{r}/r^2} = \nabla^2\frac{1}{r} = -4\pi\,\delta^3(\vec{r})$$