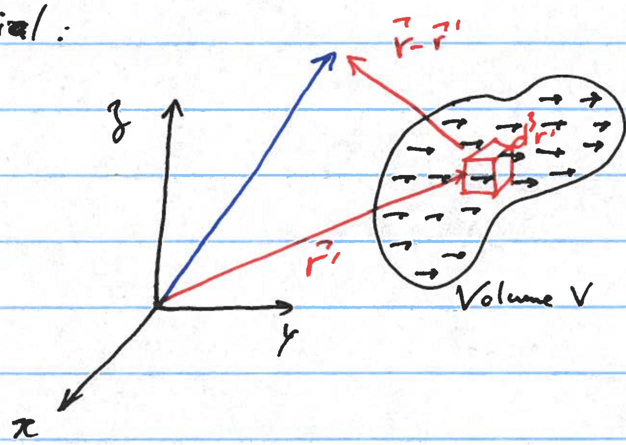
Monday, April 3, 2023

Electrostatics in Matter [chpt 4.2]

We consider the potential $V(\vec{r})$ produced by a polarized material:



$\vec{P}(\vec{r}')$ = polarization / dipole moment per volume

potential V

Recall: For a single dipole $\vec{p}$: $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}\cdot\hat{r}}{r^2}$

For a continuous distribution of $\vec{p}$'s (i.e. our material): $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \frac{\vec{P}(\vec{r}')\cdot\widehat{(\vec{r}-\vec{r}')}\,d^3r'}{|\vec{r}-\vec{r}'|^2}$

using the relation $\vec{\nabla}_{r'}\left(\dfrac{1}{|\vec{r}-\vec{r}'|}\right) = \dfrac{\widehat{(\vec{r}-\vec{r}')}}{|\vec{r}-\vec{r}'|^2}$, one can show that

a few lines of multivariable calculus

surface charge distribution

volume charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\vec{P}(\vec{r}')\cdot\hat{n}'\,ds'}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\varepsilon_0}\int_V \frac{\left(-\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r}')\right)d^3r'}{|\vec{r}-\vec{r}'|}$$

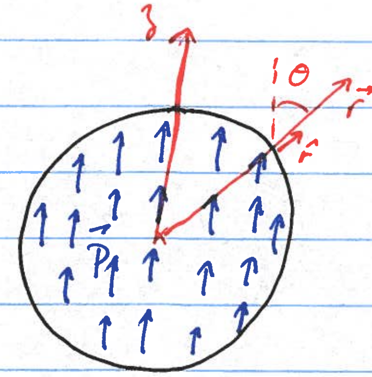
$\Rightarrow$ Thus we infer (and define):

volume bound charge: $\rho_b(\vec{r}) \equiv -\vec{\nabla}\cdot\vec{P}(\vec{r})$

surface bound charge: $\sigma_b(\vec{r}) \equiv \vec{P}(\vec{r})\cdot\hat{n}$

$\hat{n}$ = normal ~~vec~~ unit vector pointing out of surface

Example: Calculate the potential of a uniformly polarized dielectric sphere (radius $= R$) with polarization $\vec{P} = cst.$



Solution: Pick $\vec{P} = P\hat{z}$

$$\vec{\nabla}\cdot\vec{P} = 0 \quad \Rightarrow \quad \boxed{\rho_b(\vec{r}) = 0}$$

$$\vec{P}\cdot\underset{\hat{r}}{\hat{n}} = P\hat{z}\cdot\hat{r} = P\cos\theta$$

$$\Rightarrow \boxed{\sigma_b(\vec{r}) = P\cos\theta}$$

Thus $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma_b(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^2r' = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{P\cos\theta'}{|\vec{r}-\vec{r}\,'|}R^2\sin\theta' d\theta' d\phi'$$

or use separation of variables !!

→ We already solved this problem and got (see lecture 14): [Wednesday, March 22]

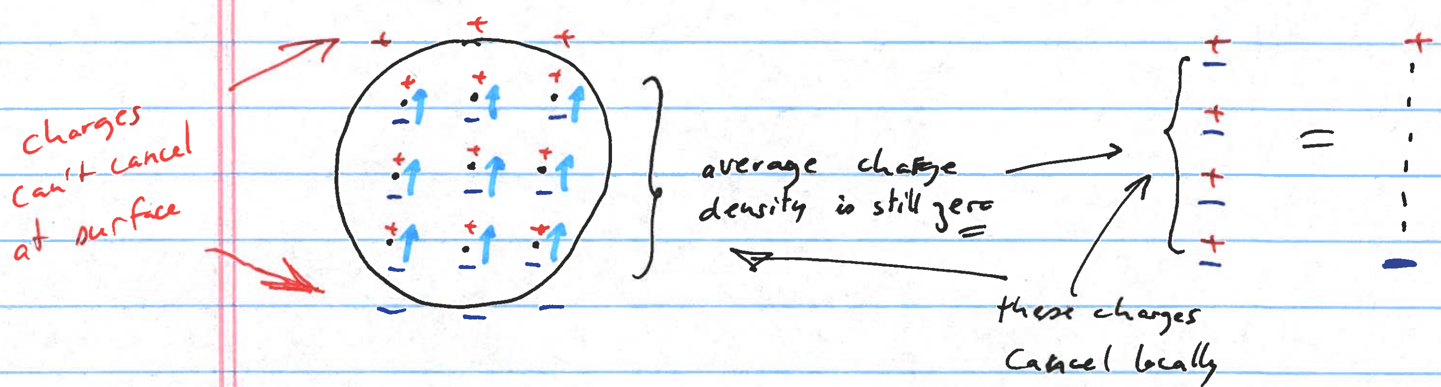
$$V_{in}(\vec{r}) = \frac{\overset{P}{\sigma_0}}{3\varepsilon_0}\underbrace{r\cos\theta}_{z} \qquad \text{for } r \le R$$

$$V_{out}(\vec{r}) = \frac{\overset{P}{\sigma_0} R^3}{3\varepsilon_0}\frac{\cos\theta}{r^2} \qquad \text{for } r \geq R$$

$$= \frac{1}{4\pi\varepsilon_0}\underbrace{\underbrace{\frac{4\pi R^3}{3}}_{\text{volume}}\underset{\text{polarization per volume}}{P}}_{\text{total dipole moment } = \mathbb{P}_z = \frac{4\pi R^3}{3}\times P\hat{z}}\overset{\hat{z}\cdot\hat{r}}{\cos\theta}\frac{1}{r^2}$$

$$= \frac{1}{4\pi\varepsilon}\frac{\vec{\mathbb{P}}\cdot\hat{r}}{r^2} \quad \Rightarrow \text{ it's a perfect dipole field/potential}$$

Q: Why is all the bound charge on the surface?
[see chpt 4.2.2, 4.2.3]



Gauss' law & Electric Displacement $\vec{D}$ [chpt 4.3.1]

Gauss's law always applies, thus inside a dielectric we have

"controlled" by experimenter

$$\vec{\nabla}\cdot \underbrace{\vec{E}_{total}(\vec{r})}_{\vec{E}} = \frac{\rho_{total}(\vec{r})}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\rho_{free} + \rho_b\right)$$

from material $-\vec{\nabla}\cdot\vec{P}$

$$\Rightarrow \varepsilon_0\, \vec{\nabla}\cdot\vec{E} = \rho_{free} - \vec{\nabla}\cdot\vec{P}$$

$$\Leftrightarrow \vec{\nabla}\cdot\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_{free}$$

We define the electric displacement $\vec{D}$ as $\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}}$

↳ $\vec{D}$ obeys "Gauss's law": $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_{free}}$

$$\Leftrightarrow \boxed{\oint_S \vec{D}\cdot d\vec{s} = Q_{free,\,enclosed}}$$

$\Rightarrow$ "Electrostatics as usual", but with $\vec{D}$ instead of $\vec{E}$, $\varepsilon_0 \to 1$, and ρ_{free} instead of ρ.

$$\vec{\nabla}\times\vec{D} = \varepsilon_0 \underbrace{\vec{\nabla}\times\vec{E}}_{=0} + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P} \neq 0 \Rightarrow$$ not quite electrostatics as usual.

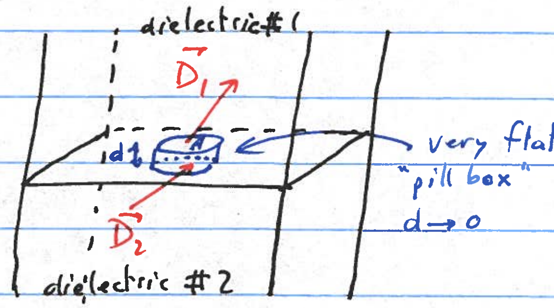
especially at a surface!!

note: If you use only "Gauss's law" + symmetry, then getting $\vec{D}$ is straightforward.

There is no displacement potential.

Boundary Conditions [chpt 4.3.3]

Consider a dielectric interface



$$\oint_S \vec{D}\cdot d\vec{s} = Q_{free,enclosed}$$

$$\Rightarrow A\vec{D}_1\cdot\hat{n}_1 + A\vec{D}_2\cdot\underbrace{\hat{n}_2}_{-\hat{n}_1} = \sigma_{free}A \Rightarrow \boxed{\left(\vec{D}_1-\vec{D}_2\right)_\perp = \underbrace{\sigma_{free}}}$$

often $\sigma_{free}=0$

note: $\left(\vec{E}_1-\vec{E}_2\right)_\perp = \dfrac{\sigma_f+\sigma_b}{\varepsilon_0}$

Also, at a charged surface $\hat{n}\times\Delta\vec{E}=0 \Leftrightarrow \hat{n}_1\times\left(\vec{E}_1-\vec{E}_2\right)=0$

(equivalent to $\Delta\vec{E}_{\parallel}=0$, i.e. $\left(\vec{E}_1-\vec{E}_2\right)_{\parallel}=0$)

↳ Then $\hat{n}_1\times\left(\vec{D}_1-\vec{D}_2\right)$

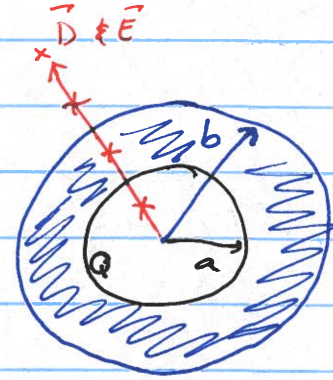
$$= \hat{n}_1\times\left[\varepsilon_0\left(\vec{E}_1-\vec{E}_2\right)+\vec{P}_1-\vec{P}_2\right] = \hat{n}_1\times\left(\vec{P}_1-\vec{P}_2\right)$$

$$\Rightarrow \boxed{\left(\vec{D}_1-\vec{D}_2\right)_{\parallel} = \left(\vec{P}_1-\vec{P}_2\right)_{\parallel}}$$

Example: Consider a hollow sphere of radius $\underline{a}$ with a uniform charge Q spread over it's surface and surrounded by a thick spherical shell of insulating plastic of radius $\underline{b}$ (thickness $= b - a$)

$\longrightarrow$ Calculate $\vec{D}$ & $\vec{E}$

note: $Q = Q_{free}$



for $r<a$

$\vec{E}=0$ from Gauss's law [$Q_{enclosed}=0$]

$\vec{P}=0$ [vacuum/air]

$\Rightarrow$ $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = 0$

for $r>b$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} \quad \text{from Gauss's law}$$

$\vec{P} = 0$ (vacuum/air)

$$\Rightarrow \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \frac{1}{4\pi}\frac{Q}{r^2}\hat{r}$$

Alternatively, "Gauss's law" for $\vec{D}$ gives: $\oint_S \vec{D}\cdot d\vec{s} = Q_{free,\,enclosed}$

$$= Q$$

$$\Rightarrow 4\pi r^2 D = Q \Rightarrow \vec{D} = \frac{1}{4\pi}\frac{Q}{r^2}\hat{r}$$

for $a<r<b$

"Gauss's law" for $\vec{D}$ gives: $\oint_S \vec{D}\cdot d\vec{s} = Q$

$$\Rightarrow \vec{D} = \frac{1}{4\pi}\frac{Q}{r^2}\hat{r}$$

Since we don't know $\vec{P}$ (yet), we cannot get $\vec{E}$!