Wednesday, April 5, 2023

Linear Dielectrics

In a linear dielectric (i.e. most materials for small $\vec{E}$), the polarization (dipole moment per volume) is given by

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad \text{where } \chi_e = \text{electric susceptibility}$$

note 1: For large $\vec{E}$, there is a deviation from linear behavior:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} + \underbrace{\varepsilon_0 \chi_e^{(2)} |\vec{E}|^2 \hat{e}}_{\substack{\text{responsible for} \\ 2^{nd} \text{ harmonic generation} \\ (\text{e.g. } 1064\text{ nm} \to 532\text{ nm})}} + \underbrace{\varepsilon_0 \chi_e^{(3)} |\vec{E}|^3 \hat{e}}_{\substack{\text{responsible for} \\ \text{4-wave mixing}}} + \ldots$$

note 2: In many crystals $\chi_e \to \chi_{e_{ij}}$ tensor susceptibility (still a linear dielectric)

Thus $P_i = \varepsilon_0 \chi_{e_{ij}} E_j$

$$\Leftrightarrow \quad \vec{P} = \varepsilon_0 \underbrace{[\chi_e]}_{\chi_{e_{ij}} \text{ matrix}} \vec{E}$$

If $\vec{P} = \varepsilon_0 \chi_e \vec{E}$, then $\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$

$$\Leftrightarrow \quad \vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}$$

$$\Rightarrow \quad \boxed{\vec{D} = \varepsilon \vec{E} \quad \text{with } \varepsilon = \varepsilon_0 (1 + \chi_e)}$$

$\Rightarrow$ "Gauss's law": $\boxed{\vec{\nabla} \cdot \vec{E} = \dfrac{\rho_{free}}{\varepsilon}}$

Dielectric constant ε_r (or relative permittivity)

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_o = \underbrace{K}_{\text{common notation}}$$

Examples:

$$\varepsilon_{vacuum} = 1$$

$$\varepsilon_{air} \simeq 1$$

$$\varepsilon_{glass} \simeq \underset{\text{BK7}}{2.3} - \underset{\text{Pyrex}}{4.8}$$

$$\varepsilon_{diamond} \simeq 5.7 - 5.9$$

$$\varepsilon_{teflon} \simeq 2.1, \quad \varepsilon_{polypropylene} \simeq 2.3$$

$$\varepsilon_{Si} \simeq 11.7$$

$$\varepsilon_{AlN} \simeq 8.9 - 9$$

$$\varepsilon_{GaAs} \simeq 12.4$$

⚠

$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P} = \vec{\nabla} \times (\varepsilon_0 \chi_e \vec{E})$$

$$= \varepsilon_0 \chi_e \cancelto{0}{(\vec{\nabla} \times \vec{E})} - \varepsilon_0 \vec{E} \times \vec{\nabla} \chi_e$$

$$= -\varepsilon \vec{E} \times \underbrace{\vec{\nabla} \chi_e}_{=0 \text{ in bulk of material}}$$

$\neq 0$ across an interface

Boundary Conditions

$$(\vec{D}_1 - \vec{D}_2)_\perp = \underset{\text{free surface charge}}{\sigma_{free}}$$

for a linear dielectric $(\vec{D} = \varepsilon \vec{E})$: $\varepsilon_1 \vec{E}_1 - \varepsilon_2 \vec{E}_2 = \sigma_f$

$\vec{E} = -\vec{\nabla} V$, so we get $\boxed{\varepsilon_1 \frac{\partial V}{\partial n} - \varepsilon_2 \frac{\partial V}{\partial n} = -\sigma_f}$ $\hat{n}$ points from "2" into "1"

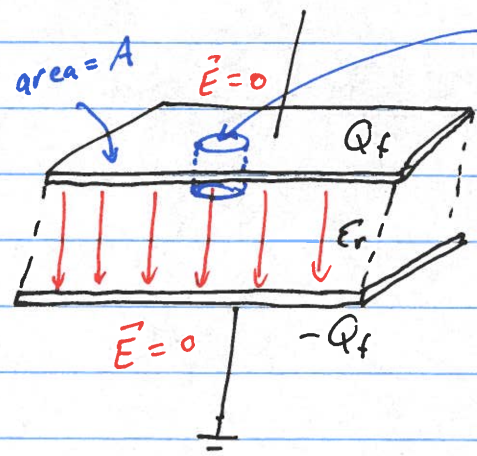
$\Delta V = -\int \vec{E}\cdot d\vec{l} \Rightarrow$ V is continuous across a boundary

↳ at boundary $V_1\Big|_{boundary} = V_2\Big|_{boundary}$

See example 4.7 in book for separation of variables + linear dielectrics

Capacitors & dielectrics

example: parallel plate capacitor with a linear dielectric between the plates



Gaussian pillbox: $EA = \dfrac{Q_f}{\varepsilon}$

note: ignore/neglect fringing field

$\vec{\nabla}\cdot\vec{E} = \dfrac{\rho_f}{\varepsilon}$

$$\vec{E}_{in} = -\frac{1}{\varepsilon}\frac{Q_f}{A}\hat{z} = -\frac{\sigma_f}{\varepsilon}\hat{z} \Rightarrow \Delta V = -\int_{\text{bottom plate}}^{\text{top plate}} \vec{E}\cdot d\vec{z}$$

(with $\varepsilon = \varepsilon_r\varepsilon_0$) (reference, $V=0$)

$$= -E_{in}d$$

$$= \frac{1}{\varepsilon}\frac{Q_f\, d}{A}$$

$$\Rightarrow \Delta V = \frac{d}{\varepsilon A}Q_f \Leftrightarrow \frac{\varepsilon A}{d} = \frac{Q_f}{\Delta V} = \underset{\uparrow\ \text{capacitance}}{C}$$

$$\Rightarrow \boxed{C_{dielectric} = \frac{\varepsilon A}{d}}$$

note: $$\frac{C_{dielectric}}{C_{vacuum}} = \frac{\varepsilon\, A/d}{\varepsilon_0\, A/d} = \varepsilon_r$$

$$\Rightarrow \boxed{C_{dielectric} = \varepsilon_r\, C_{vacuum}}$$

$\Rightarrow$ adding the dielectric increases the capacitance.

Electrostatic Energy in Dielectric systems

In vacuum: $$W = \frac{\varepsilon_0}{2} \int_V \vec{E}^2\, d^3r$$

In a dielectric: $$W = \frac{1}{2} \int_V \underbrace{\vec{D}}_{\varepsilon \vec{E}} \cdot \vec{E}\, d^3r$$

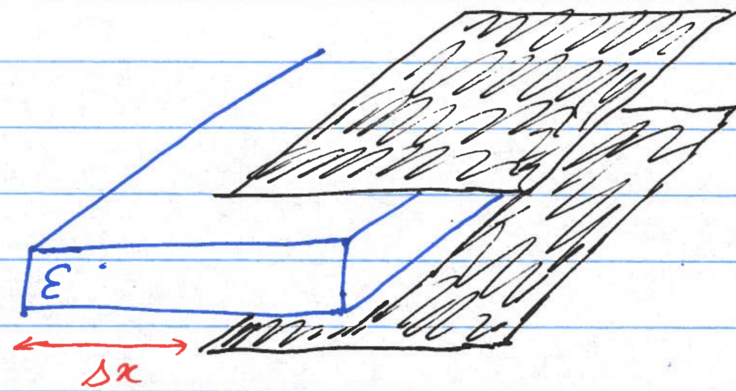
$$\Rightarrow W = \frac{\varepsilon}{2} \int \vec{E}^2\, d^3r$$

Example: Force on a dielectric in a capacitor

reminder: $$W = \frac{1}{2} C V^2 = \frac{1}{2} C \left(\frac{Q}{C}\right)^2 = \frac{1}{2} \frac{Q^2}{C}$$

$C = \frac{Q}{V} \Leftrightarrow V = Q/C$

Consider a capacitor with a sliding dielectric



$$C_\varepsilon = \varepsilon_r C_{\varepsilon_0} \quad \Rightarrow \quad C_\varepsilon > C_{\varepsilon_0}$$

$\varepsilon_r > 1$

$$\Rightarrow \boxed{W_\varepsilon < W_{\varepsilon_0}}$$

for constant Q

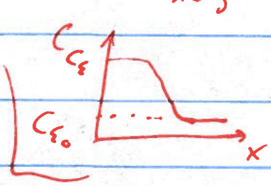
(different approach needed for constant V)

Capacitor with dielectric has lower energy than capacitor without dielectric

The capacitor will "suck in" the dielectric to lower the energy of the system.

$$\text{Force} = F = -\frac{dW}{dx} = -\frac{d}{dx}\left[\frac{1}{2}\frac{Q^2}{C(x)}\right]$$

$$= -\frac{1}{2}Q^2 \underbrace{\frac{d}{dx}\frac{1}{C(x)}}_{-\frac{1}{C^2}\frac{dC(x)}{dx}} = \frac{1}{2}\frac{Q^2}{C^2}\underbrace{\frac{d}{dx}C(x)}_{\text{negative}}$$



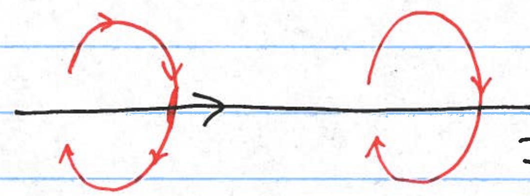
$\Rightarrow$ force is in direction opposite to increasing x

$\Rightarrow$ dielectric is "sucked in"

Magnetostatics [chpt 5]

Magnetic fields are generated by moving charges, i.e. currents.

Ex: Right hand rule gives the direction of B-field lines



I ← direction of conventional current (positive charges)

put your thumb along direction of I
↳ fingers give direction of B-field

magnetic force on a moving charge

$$\vec{F}_{magnetic} = q\vec{v}\times\vec{B}$$

⟹ force is $\perp$ to motion $\vec{v}$ and $\vec{B}$

Force [Newtons]

charge [Coulombs]

velocity [m/s]

magnetic field [Tesla]

note: 1 Tesla = 10^4 Gauss

Examples:

B-field of the Earth: $B_{\substack{\text{Earth}\\ \text{surface}}} \simeq 0.3-0.5$ G

Refrigerator/house magnet $B \sim 100$ G (right at surface)

$B_{\substack{\text{Jupiter}\\ \text{equator}}} \simeq 4$ G , $B_{\substack{\text{sun}\\ \text{surface}}} \simeq 1$ G , $B_{\text{sunspot}} \simeq 3000$ G near Solar System

Galactic center region

$B_{Milky way} \simeq \underset{\uparrow}{6} - \underset{\uparrow}{40}\ \mu G \simeq 10^{-5}\ G$, $B_{\substack{neutron\\star}} = 10^{8-15}\ G$

$B_{\substack{medical\\MRI}} \simeq 1\ T = 10{,}000\ G$, $B_{\substack{NMR\ Lab\\W\&M}} \simeq 17\ T \simeq 170{,}000\ G$