Monday, April 10, 2023

Lorentz Force Law

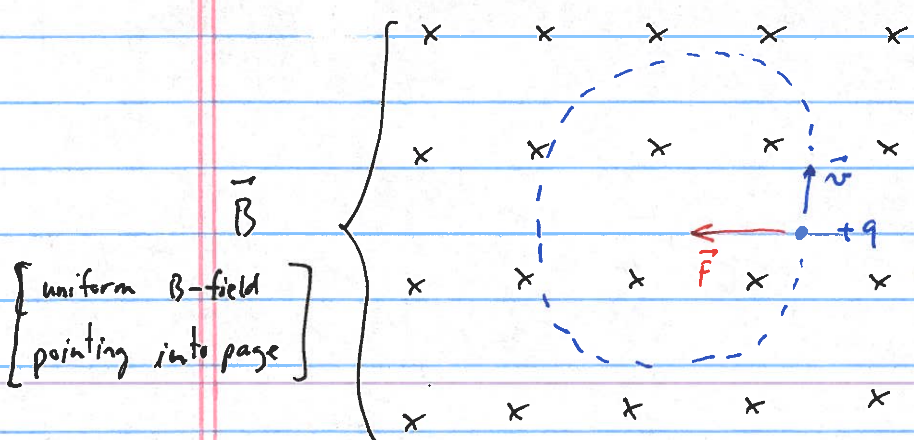
Force on a point charge q, with velocity $\vec{v}$, in an electric field $\vec{E}$ and a magnetic field $\vec{B}$:

$$\boxed{\vec{F}_{EM} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)}$$

(also works for time-dependent EM fields.)

Cyclotron Motion

(no E-field) $\vec{v} \perp \vec{B}$



$\Rightarrow$ moving charges tend to travel in circular orbits in a magnetic field

$$F = qvB \Rightarrow a = \frac{qvB}{m}$$

For circular motion: $a_c = \dfrac{v^2}{R}$ ← radius of motion

$$\Rightarrow \frac{qvB}{m} = \frac{v^2}{R} \iff \boxed{R = \frac{mv}{qB}}$$

time to complete 1 orbit : $T = \frac{2\pi R}{v} = \frac{2\pi}{\cancel{v}} \frac{m\cancel{v}}{qB} = \frac{2\pi m}{qB}$

$\Rightarrow$ frequency $= f = \frac{1}{T} = \frac{qB}{2\pi m}$ (Hz)

$\Rightarrow$ $\boxed{\text{cyclotron frequency} = \omega_c = \frac{qB}{m}}$ (rads/s)

<u>Magnetic Work</u> : <u>Magnetic forces do NO work</u>

(quantum exceptions)

<u>proof</u>: Work $= W = \vec{F}\cdot\Delta\vec{x} = \int_{\text{path of travel}} \vec{F}\cdot d\vec{x}$

$= \int \vec{F}_{mag}\cdot d\vec{x}$

$= \int q(\vec{v}\times\vec{B})\cdot \underbrace{d\vec{x}}_{\vec{v}dt}$

$= q\int \underbrace{\underbrace{(\vec{v}\times\vec{B})}_{\perp \text{ to } \vec{v} \,\&\, \vec{B}}\cdot\vec{v}}_{=0}\,dt = 0$

$\Rightarrow$ $\boxed{W_{mag} = 0}$

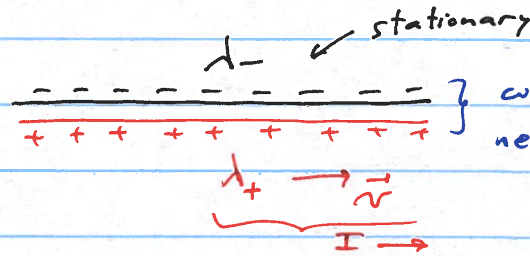
<u>Note</u>: Charge q is independent of reference frame.
[different from mass]

[see Griffiths, example 5.3 for a good discussion]

Current & Current Density

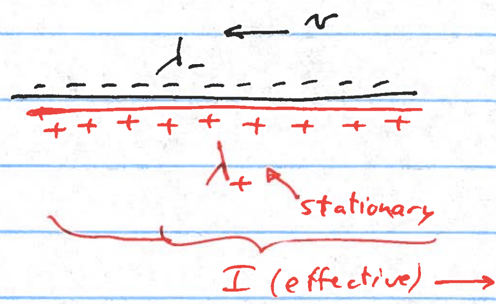
Current in a wire

model of a wire :
(conventional current)



wire is always neutral, since λ_- cancel λ_+
i.e. $\lambda_- = -\lambda_+$

real current (e^-) :



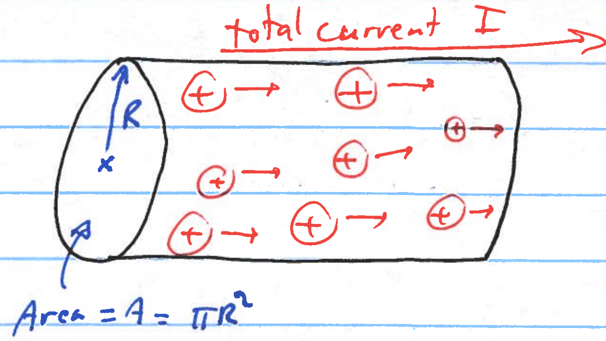
↳ Current $I = \lambda v$ $\qquad \left[\frac{C}{m} \times m/s\right] = \left[C/s\right]$

note: 1 Ampère = 1 A = 1 C/s = $6.24 \times 10^{18}\ e^-/s$

Current Density $\vec{J}$

Consider a current of charges flowing uniformly through a wire (or charged particles in particle beam at an accelerator).

J = Current density = current per unit Area



$$\Rightarrow \boxed{J = \frac{I}{\text{wire cross-section Area}} = \frac{I}{\pi R^2}}$$

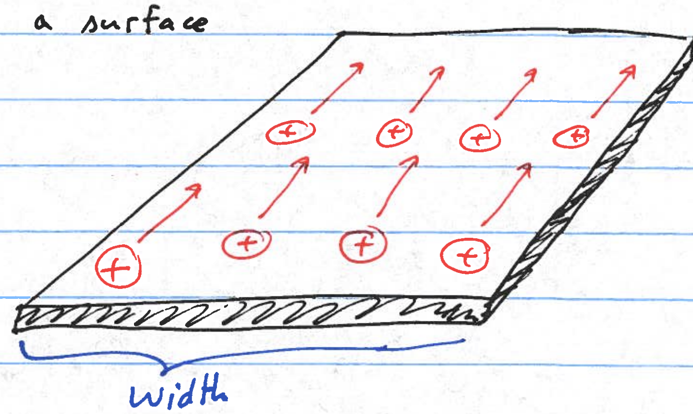
Formal vector definition: $\boxed{\vec{J} = \rho \vec{v}}$

$\frac{C}{m^2 \cdot s}$ $\frac{C}{m^3}$ m/s

Surface Current Density $\vec{K}$

Current confined to a surface

$$\boxed{\vec{K} = \sigma \vec{v}}$$



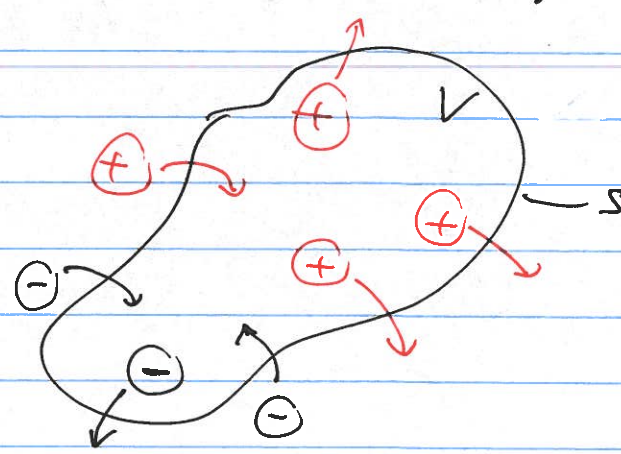
$$|\vec{K}| = \frac{I}{\text{width}}$$

$$= [A/m]$$

$$= \left[\frac{C}{m \cdot s}\right]$$

Continuity Equation (conservation of charge)

Consider a volume V with surface S with charges flowing out (in) of it.



total current flowing out

$$= I_{out} = \int_S \rho \vec{v} \cdot d\vec{s}$$

$$= \int_S \vec{J} \cdot d\vec{s}$$

= current flowing across surface

I_{out} is also the total change in charge inside V per unit time.

$$I_{out} = \underbrace{-\frac{d}{dt}\int_V \rho\, d^3r}$$

negative for charge flowing out

Thus,

$$-\frac{d}{dt}\int_V \rho\, d^3r = \underbrace{\int_S \vec{J}\cdot d\vec{s}}$$

Apply divergence theorem

$$\int_V \left(-\frac{d}{dt}\rho\right) d^3r = \int_V (\vec{\nabla}\cdot\vec{J})\, d^3r$$

Since this expression/equality is valid for any volume V, including an infinitesimal one, then the integrands must be equal:

$$\boxed{-\frac{d\rho}{dt} = \vec{\nabla}\cdot\vec{J} \iff \frac{d}{dt}\rho + \vec{\nabla}\cdot\vec{J}}$$

Continuity equation for local conservation of charge

Note 1: if $\frac{d\rho}{dt} = 0$, then $\vec{\nabla}\cdot\vec{J} = 0 \leftarrow$ magnetostatics

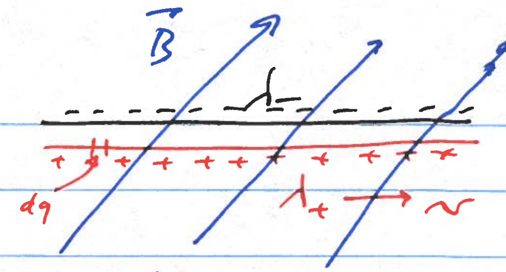
Note 2: local conservation of energy: $-\frac{\partial}{\partial t}$ Energy density $= \vec{\nabla}\cdot$ Power density

local conservation of momentum: $-\frac{\partial}{\partial t}\vec{p} = \vec{\nabla}\cdot \overleftrightarrow{\text{Stress tensor}}$

$\vec{p}$ ↑ momentum density; Stress tensor ↑ "external forces"

[see chpt 8]

Force on a current



$$\vec{F}_{magnetic} = \int_{\text{wire length}} (\vec{v} \times \vec{B})\, dq = \int_{\text{wire length}} (\vec{v} \times \vec{B})\, \lambda\, dl$$

$$= \int \big(\underbrace{\lambda \vec{v}}_{\text{"}\vec{I}\text{"}} \times \vec{B}\big)\, dl = \int \underbrace{I}\, d\vec{l} \times \vec{B} = I \int d\vec{l} \times \vec{B}$$

current I is constant along wire

$$\Rightarrow \boxed{\vec{F}_{magnetic} = I \int_{\text{wire length}} d\vec{l} \times \vec{B}}$$

Biot-Savart Law

Magnetic field due to an infinitesimal current segment $I d\vec{l}$:

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^2} \quad \text{with } \mu_0 = 4\pi \times 10^{-7}\ N/A^2$$

= permeability of free space

$$\boxed{\begin{aligned} \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \widehat{(\vec{r}-\vec{r}\,')}\, dl'}{|\vec{r}-\vec{r}\,'|^2} \\ &= \frac{\mu_0 I}{4\pi} \int_{\text{current or wire length}} \frac{d\vec{l}\,' \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2} \end{aligned}}$$