Wednesday, April 12, 2023

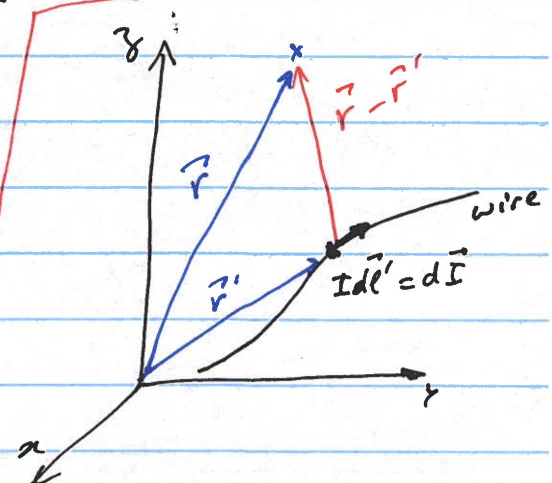
Biot-Savart Law [chpt 5.2]

The magnetic field due to an infinitesimal current segment $I d\vec{\ell}$ is

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \hat{r}}{r^2} \quad \text{with } \mu_0 = 4\pi \times 10^{-7}\, N/A^2$$

$$\Rightarrow \boxed{\vec{B} = \int_{\text{wire}} dB(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{\vec{I} \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\, d\ell' = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\vec{\ell}\,' \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}}$$

Biot-Savart Law

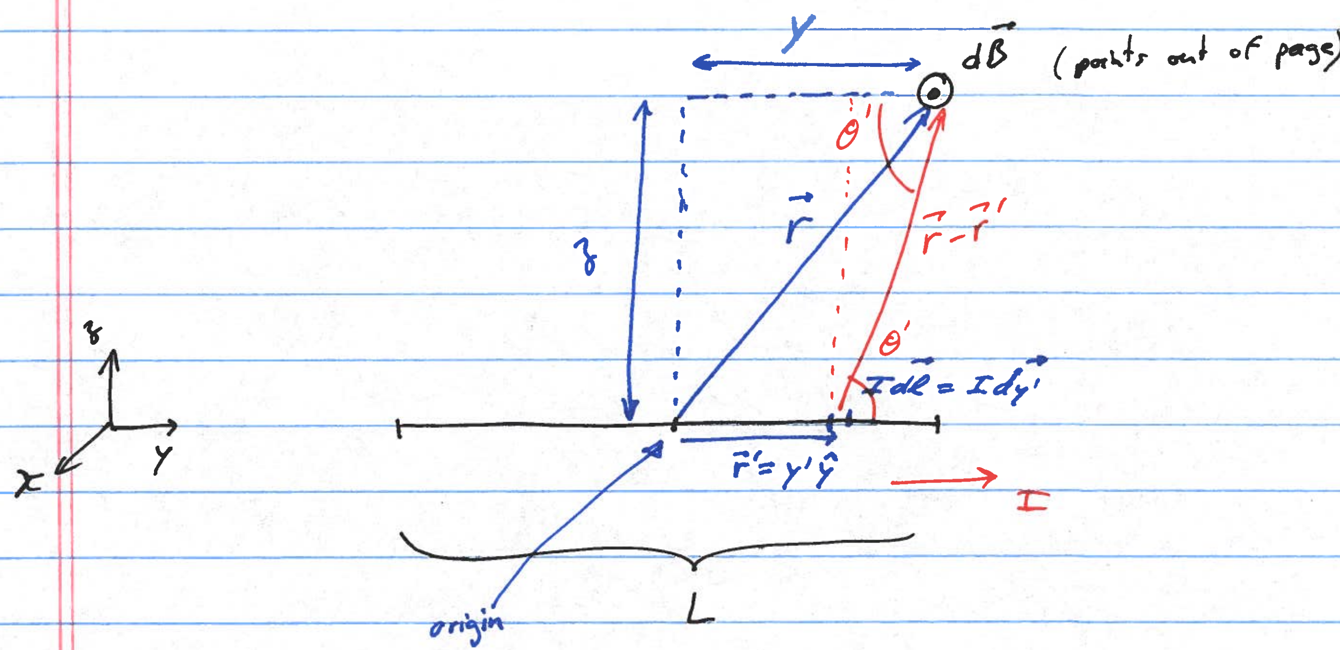


For a volume current:
$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}\,') \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\, d^3r'}$$

$\vec{J}$ = current density

For a surface current:
$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}(\vec{r}\,') \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}}$$

Example: magnetic field from a wire segment



$$d\vec{B} = \frac{\mu_0}{4\pi} I \underbrace{\frac{d\vec{y}\,' \times \widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}}$$

note: $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta \, (-\hat{x})$

$$\frac{dy' \sin\theta' \, \hat{x}}{|\vec{r}-\vec{r}\,'|^2} = \frac{dy' \sin\theta' \, \hat{x}}{(y-y')^2 + z^2}$$

$$= \frac{dy' \, \hat{x}}{(y-y')^2 + z^2} \, \frac{z}{\sqrt{(y-y')^2 + z^2}} = \frac{z \, \hat{x} \, dy'}{\left[(y-y')^2 + z^2\right]^{3/2}}$$

$$\Rightarrow \vec{B} = \int_{-L/2}^{L/2} \frac{\mu_0 I}{4\pi} \frac{z \, \hat{x} \, dy'}{\left[(y-y')^2 + z^2\right]^{3/2}} = \frac{\mu_0 I}{4\pi} z \hat{x} \int_{-L/2}^{L/2} \frac{dy'}{\left[(y-y')^2 + z^2\right]^{3/2}}$$

substitution: $u = y - y' \Rightarrow du = -dy'$ $\Big|$ $= -\frac{\mu_0 I}{4\pi} z \hat{x} \int_{y'=-L/2}^{y'=L/2} \frac{du}{(u^2 + z^2)^{3/2}}$

$$\dots \xrightarrow[\substack{\text{trigonometric}\\ \text{substitution}\\ \text{see Lectures}\\ \text{Feb. 8, 2023}}]{} = -\frac{\mu_0}{4\pi} I \cancel{z}\,\hat{x} \; \frac{u}{z^{\cancel{2}}\sqrt{u^2+z^2}} \Bigg|_{y'=-L/2}^{y'=L/2}$$

$$\begin{bmatrix} u = z\tan\varphi \\ du = z\sec^2\varphi\, d\varphi \end{bmatrix}$$

$$= -\frac{\mu_0}{4\pi} I\,\hat{x}\; \frac{y-y'}{z\sqrt{(y-y')^2+z^2}}\Bigg|_{y'=-L/2}^{L/2}$$

$$= -\frac{\mu_0 I}{4\pi}\frac{\hat{x}}{z}\left\{ \frac{y-(L/2)}{\sqrt{(y-L/2)^2+z^2}} - \frac{y+(L/2)}{\sqrt{(y+\frac{L}{2})^2+z^2}} \right\}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi}\frac{1}{z}\left\{ \frac{y+L/2}{\sqrt{z^2+(y+L/2)^2}} - \frac{y-L/2}{\sqrt{z^2+(y-L/2)^2}} \right\}$$

<u>note</u>: $\lim\limits_{\substack{L\to\infty \\ \text{(infinite wire)}}} \vec{B} = \frac{\mu_0 I}{4\pi}\frac{1}{z}(1+1) \xrightarrow[\text{generalize}]{} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\phi}$

Superposition Principle

Magnetic fields obey the superposition principle:

$$\vec{B}_{total}(\vec{r}) = \underbrace{\vec{B}_1(\vec{r})}_{\text{from Source 1}} + \underbrace{\vec{B}_2(\vec{r})}_{\text{from source 2}} + \dots$$

Divergence of $\vec{B}$

$$\vec{\nabla}_r \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \underbrace{\vec{\nabla}_r \cdot \left[\vec{J}(\vec{r}\,') \times \frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\right]}_{\frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\cdot\underbrace{\left(\vec{\nabla}_r \times \vec{J}(\vec{r}\,')\right)}_{=0} - \vec{J}(\vec{r}\,')\cdot\underbrace{\left(\vec{\nabla}_r \times \frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\right)}_{=0}} d^3r'$$

rule 6 from inside front cover of book

no "r" dependence

(does not have any rotational circulation ... see problem 1.63)

$$= 0$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

"no magnetic monopoles" law

Curl of $\vec{B}$

$$\vec{\nabla}_r \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \underbrace{\vec{\nabla}_r \times \left[\vec{J}(\vec{r}\,') \times \frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\right]}\, d^3r'$$

rule 8 from inside front cover of book

$$= \underbrace{\cancel{\left(\frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\cdot\vec{\nabla}_r\right)\vec{J}(\vec{r}\,')}}_{=0} - \left(\vec{J}(\vec{r}\,')\cdot\vec{\nabla}_r\right)\frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}$$

$$+ \vec{J}(\vec{r}\,')\underbrace{\left(\vec{\nabla}_r\cdot\frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\right)}_{4\pi\,\delta^3(\vec{r}-\vec{r}\,')} - \frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\underbrace{\cancel{\left(\vec{\nabla}_r\cdot\vec{J}(\vec{r}\,')\right)}}_{=0}$$

$$\Rightarrow \vec{\nabla}_r \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left\{ \underbrace{\int_V 4\pi\,\delta^3(\vec{r}-\vec{r}\,')\,\vec{J}(\vec{r}\,')\,d^3r'}_{=\,4\pi\vec{J}(\vec{r})} - \underbrace{\int_V \left(\vec{J}(\vec{r}\,')\cdot\vec{\nabla}_r\right)\frac{\widehat{(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|^2}\,d^3r'}_{=\,0} \right\}$$

convert to a surface integral involving $\vec{J}$ and $\vec{\nabla}\cdot\vec{J}$

(and $\vec{\nabla}\cdot\vec{J}=0$ for magnetostatics)

but $\vec{J}$ is generally zero on a surface for a large volume.

$= 0$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0\vec{J}(\vec{r})}$$

Ampère's Law

In integral form: $\displaystyle \int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{s} = \int_S \mu_0 \vec{J}\cdot d\vec{s}$

(integrate over an open surface S)

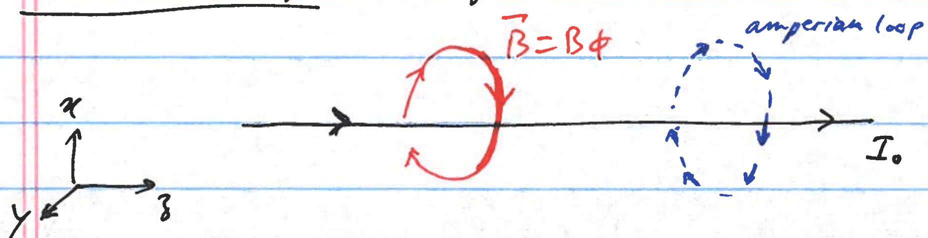
Stoke's theorem: $\displaystyle \oint_{\text{boundary of } S\,=\,\text{loop}} \vec{B}\cdot d\vec{\ell}$

$\displaystyle \mu_0 \underbrace{\int_S \vec{J}\cdot d\vec{s}}_{I_{\text{enclosed}}}$

$$\Rightarrow \boxed{\oint_{\text{loop}} \vec{B}\cdot d\vec{\ell} = \mu_0 I_{\text{enclosed}}}$$

Ampère's law in integral form

Classic Example: magnetic field of a very long wire



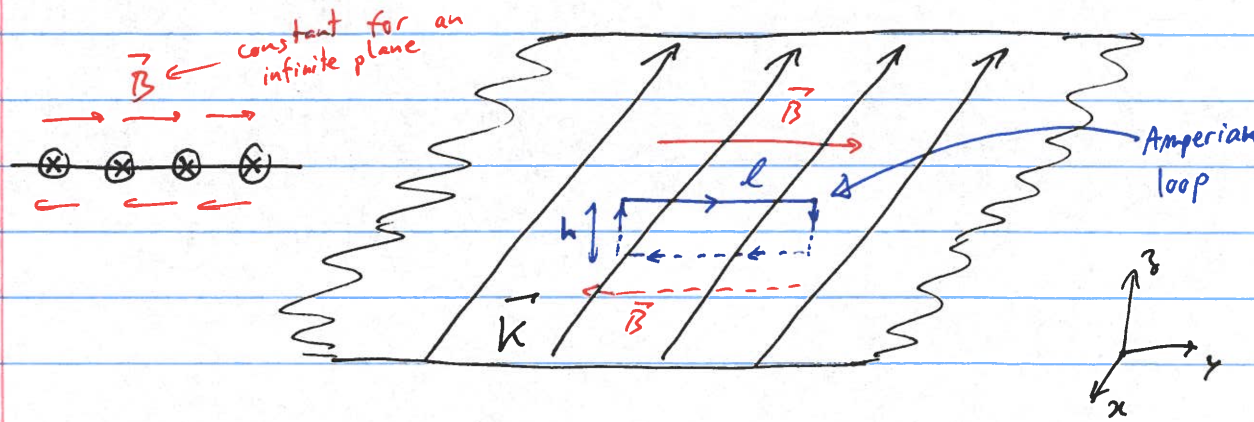
note: use right hand rule to determine if I_{enclosed} is $\begin{cases}\text{positive}\\ \text{negative}\end{cases}$

$$\oint_{loop} \vec{B}\cdot d\vec{\ell} = \mu_0 I \quad \Rightarrow \quad B\, 2\pi r = \mu_0 I$$

$$\Leftrightarrow \quad \boxed{\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\phi}}$$

⚠ Use Ampère's law whenever there is symmetry.
(otherwise, use Biot-Savart law)

Example: Planar surface current



$$\oint_{loop} \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enclosed}$$

$$\Leftrightarrow \quad \underbrace{\vec{B}\cdot\vec{l}}_{/\!/} + \underbrace{\vec{B}\cdot\vec{h}}_{=0} + \underbrace{\vec{B}\cdot\vec{l}}_{/\!/} + \underbrace{\vec{B}\cdot\vec{h}}_{=0} = \mu_0 K \ell$$

$\vec{B} \perp \vec{h}$

$$\Leftrightarrow \quad 2B\ell = \mu_0 K \ell \quad \Leftrightarrow \quad \boxed{\begin{array}{ll} \vec{B} = \frac{\mu_0 K}{2}\hat{y} & \text{above} \\ \vec{B} = -\frac{\mu_0 K}{2}\hat{y} & \text{below} \end{array}}$$

B-field is independent of height
(for an infinite plane)