Wednesday, April 19, 2023

Magnetic Vector Potential

reminder: $\vec{\nabla}\times\vec{E}=0 \implies$ potential V exists such that

$$\underset{\text{3 components}}{\vec{E}} = -\vec{\nabla}\underset{\text{1 component}}{V}$$

Question: since $\vec{\nabla}\cdot\vec{B}=0$, is there a corresponding "potential"?

Answer: Yes! The vector potential $\vec{A}(\vec{r})$: $\boxed{\underset{\text{3 components}}{\vec{B}} = \vec{\nabla}\times\underset{\text{3 components}}{\vec{A}}}$

Note: Just like $V(\vec{r})$, $\vec{A}(\vec{r})$ is not uniquely defined

$$\vec{A}'(\vec{r}) = A(\vec{r}) + \vec{\nabla}\lambda(\vec{r}) \implies \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \underbrace{\vec{\nabla}\times(\vec{\nabla}\lambda(\vec{r}))}_{=0}$$

random function (well behaved)

$$\implies \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A}$$

Coulomb gauge: In magnetostatics, one generally chooses $\vec{A}$ such that $\boxed{\vec{\nabla}\cdot\vec{A}=0}$ (definition of Coulomb gauge)

proof that one can always find a $\lambda(\vec{r})$ that will make $\vec{\nabla}\cdot\vec{A}=0$

- consider $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A}\neq 0$.
- we construct $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$.
- Then, we get $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \underbrace{\vec{\nabla}\cdot\vec{\nabla}\lambda}_{\nabla^2\lambda}$
- If we require $\vec{\nabla}\cdot\vec{A}'=0$, then $\nabla^2\lambda = \underbrace{-\vec{\nabla}\cdot\vec{A}}$

treat mathematically as a "scalar charge distribution"

$\Rightarrow$ we know that we can solve $\nabla^2 \lambda = \underbrace{-\vec{\nabla}\cdot\vec{A}}$
since it identical mathematically to Poisson's equation

think of this as " $-\frac{\rho(\vec{r})}{\epsilon_0}$ "

$$\Rightarrow \quad \lambda(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|} \qquad \left[\text{for } \vec{\nabla}\cdot\vec{A} \to 0 \text{ at } r \to \infty\right]$$

Ampère's law again

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \iff \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{A}}_{=0 \text{ in Coulomb gauge}}) - \nabla^2\vec{A}} = \mu_0 \vec{J}$$

[from inside front cover of book]

Thus in the Coulomb gauge $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$

$\Rightarrow$ it's just Poisson's equation for each component of $\vec{A}$.

All the electrostatics techniques for Laplace's equation can be used!!!

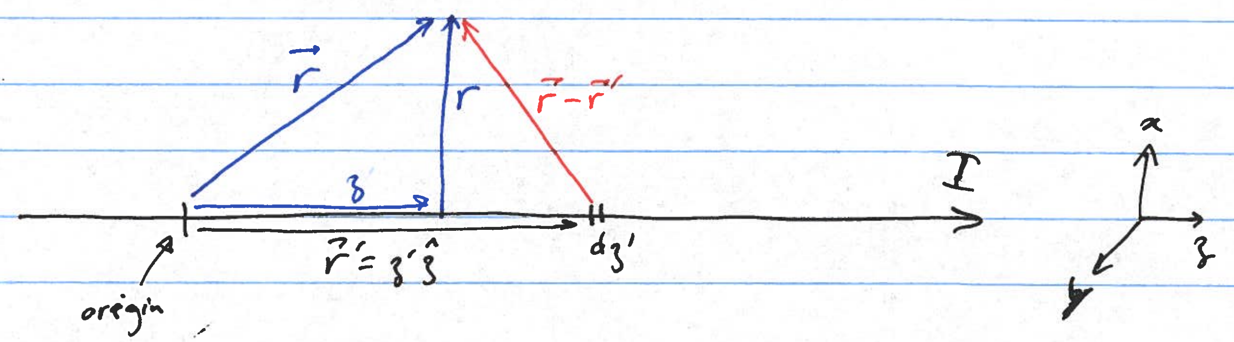
$$\Rightarrow \quad \boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}}$$

in the Coulomb gauge.

Example: Calculate $\vec{A}$ for an infinitely long thin wire with current I.

$$\vec{J}(\vec{r}\,')\,d^3r' = I\,d\vec{\ell}\,' = I\,dz'\,\hat{z}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{dz'}{|\vec{r}-\vec{r}'|}\hat{z} \qquad \text{(in Coulomb gauge)}$$



$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{z} \int_{-\infty}^{+\infty} \frac{dz'}{\sqrt{r^2+(z'-z)^2}} = \frac{\mu_0 I}{4\pi}\hat{z} \int_{-\infty}^{+\infty} \frac{du}{\sqrt{r^2+u^2}}$$

substitution
$u = z'-z$
$du = dz'$

integration by trigonometric substitution
$u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$$= \frac{\mu_0 I}{4\pi}\hat{z} \underbrace{\ln\left[u + \sqrt{r^2+u^2}\right]\Big|_{-\infty}^{+\infty}}_{\text{diverges}}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z} \lim_{L\to\infty} \ln\left[u + \sqrt{r^2+u^2}\right]\Big|_{-L}^{+L}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z} \lim_{L\to\infty} \underbrace{\ln\left\{\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right\}}_{\ln\left\{\frac{1+\sqrt{1+(r/L)^2}}{-1+\sqrt{1+(r/L)^2}}\right\}}$$

$$= \ln\left[1+\underbrace{\sqrt{1+(r/L)^2}}_{1+\frac{1}{2}\left(\frac{r}{L}\right)^2}\right] - \ln\left[-1+\underbrace{\sqrt{1+(r/L)^2}}_{1+\frac{1}{2}\left(\frac{r}{L}\right)^2}\right]$$

$$\simeq \ln\left[2 + \frac{1}{2}\left(\frac{r}{L}\right)^2 + \cdots\right] - \ln\left[\frac{1}{2}\left(\frac{r}{L}\right)^2 + \cdots\right]$$

Thus,

$$\vec{A} \simeq \frac{\mu_0 I \hat{z}}{4\pi} \lim_{L\to\infty} \left\{ \underbrace{\ln\left[2 + \frac{1}{2}\left(\frac{r}{L}\right)^2 + \cdots\right]}_{\simeq \ln(2)} - \underbrace{\left(\ln\frac{1}{2} + \ln\left(\frac{r}{L}\right)^2 + \cdots\right)}_{\left(-\ln 2 + 2\ln\left(\frac{r}{L}\right)\right)} \right\}$$

$$= \left[-\ln 2 + 2\ln(r) - 2\ln(L)\right]$$

$$\simeq \frac{\mu_0 I \hat{z}}{4\pi} \lim_{L\to\infty} \underbrace{2\ln(2) + 2\ln(L)}_{\text{diverging constant}} - 2\ln(r)$$

diverging constant (independent of x, y, z)

does not affect $\vec{B} = \vec{\nabla}\times\vec{A}$

ignore!

$$\Rightarrow \boxed{\vec{A}(\vec{r}) = -\frac{\mu_0 I}{2\pi}\ln(r)\,\hat{z} + cst\,\hat{z}}$$

<u>note 1:</u> $\vec{A}$ is not used that much in ~~e~~ magnetostatics, but it is very useful for describing time-dependent electric & magnetic potentials.

<u>note 2:</u> Quantization of the EM field is often done with $\vec{A}$.

becomes an operator

(harmonic oscillator $a^{\dagger}$ & a)

Multipole expansion of $\vec{A}$

In the Coulomb gauge:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \underbrace{\frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

(for $r > r'$)

α: angle between $\vec{r}$ & $\vec{r}'$

$\cos\alpha = \hat{r}\cdot\hat{r}'$

$$= \frac{\mu_0 I}{4\pi}\Bigg[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{\substack{\text{monopole} \\ =0}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha \, d\vec{l}'}_{\text{dipole term}} + \underbrace{\frac{1}{r^3}\oint r'^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)d\vec{l}'}_{\text{quadrupole term}} + \ldots\Bigg]$$

since $\oint d\vec{l}' = 0$

Magnetic dipole

At large distances, the magnetic dipole term typically dominates.

↳ the character of most magnetic fields is dipole-like.

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \underbrace{\oint (\vec{r}'\cdot\hat{r})\, d\vec{l}'}_{= \left(\int_{\substack{\text{surface} \\ \text{of loop}}} d\vec{s}'\right)\times\hat{r}} = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\left(I\int_{S \text{ of loop}} d\vec{s}'\right)\times\hat{r}$$

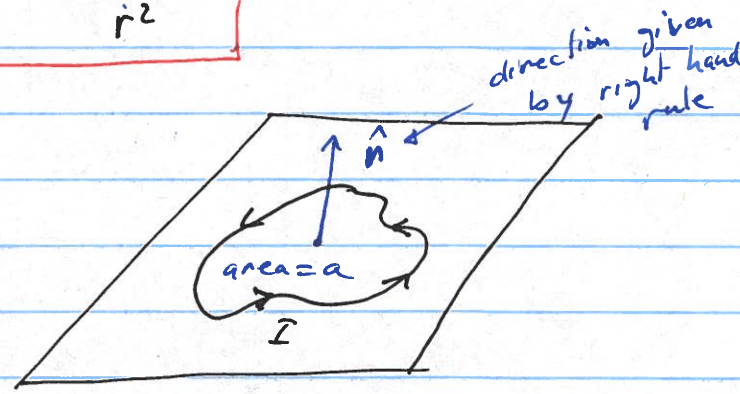
see problem 1.62 (problem set #4)

If we define the <u>magnetic moment</u> as

$$\vec{m} = I \underbrace{\int_{S \text{ of loop}} d\vec{s}\,'}_{\text{vector area} = \vec{a}} = \boxed{I\vec{a} = \vec{m}}$$

then $\boxed{\vec{A}_{dipole}(\vec{r}) = \dfrac{\mu_0}{4\pi} \dfrac{\vec{m} \times \hat{r}}{r^2}}$

For a planar current



the vector area is $\vec{a} = \text{area}\ \hat{n}$

<u>Alternate forms</u>: $\vec{A}_{dipole}(\vec{r}) = \dfrac{\mu_0}{4\pi} \dfrac{m \sin\theta}{r^2}\ \hat{\phi}$

$$\vec{B}_{dipole}(\vec{r}) = \vec{\nabla} \times \vec{A}_{dipole} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta}\right)$$

(for $\vec{m} = m\hat{z}$)

$$\Rightarrow \boxed{\vec{B}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]}$$

Compare with $\vec{E}_{dipole}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] \Rightarrow$ same form

magnetic field of an ideal magnetic dipole