Final Exam: Tuesday, May 9 at 7 pm

Distribution of topics

About 1/3 – 1/5 on 1st half of semester

About 2/3 – 4/5 on 2nd half of semester

Electrostatics

Method of Images

Separation of variables, cartesian sym.

→ Orthogonality relation

Separation of variable, spherical sym.

→ Legendre polynomials

→ Orthogonality relation

Multipole expansion

Dipole moment

→ Forces on dipoles

Polarizability of matter & dielectrics

Bound charges (surface & volume)

Electric displacement D

Linear dielectrics

capacitors

Magnetostatics

Lorentz force law

Cyclotron motion

No magnetic work

Biot-Savart law

Ampere's law (and div B = 0)

→ Solenoid, toroid, surface current

Vector potential

Multipole expansion

Dipole moment

→ Forces on dipoles

Magnetization of matter

Bound currents (surface & volume)

Auxiliary Field H

Linear magnetization

Faraday's Law & Maxwell's Equations

PHYS 401: Electricity & Magnetism 1
Monday, May 1, 2023

Formula Study Sheet

(i.e. formulas that you should know)

Gradient theorem

$$\int_{\vec{r}_a}^{\vec{r}_b} (\vec{\nabla} f) \cdot d\vec{l} = f(\vec{r}_b) - f(\vec{r}_a)$$

$path\ P$

Divergence Theorem

$$\int_V (\vec{\nabla} \cdot \vec{F})\, d^3r = \oint_{S(V)} \vec{F} \cdot d\vec{s}$$

Stokes's Theorem

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s} = \oint_{C(S)} \vec{F} \cdot d\vec{\ell}$$

Divergence of 1/r^2 - point source

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r}) \quad \& \quad \nabla^2 \frac{1}{r} = -4\pi\delta^3(\vec{r})$$

Coulomb's law

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^2}$$

Electric field of a point charge q at $\vec{r}'$

$$\vec{E}_q = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r}'|^2} (\widehat{r - r'})$$

Electric field of a charge distribution $\rho(\vec{r})$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^2} (\widehat{r - r'}) d^3r'$$

Potential of a point charge q at $\vec{r}'$

$$V_q(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r}'|}$$

Potential of a charge distribution $\rho(\vec{r})$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

Electric field and potential

$$\vec{E} = -\vec{\nabla} V \;\; \& \;\; V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{\ell}$$

Electric field of a plane of charge

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$$

Electric field across a plane of charge

$$\Delta E_\perp = \frac{\sigma}{\epsilon_0} \;\; \& \;\; \Delta E_\parallel = 0$$

Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0} \;\; \& \;\; \oint_S \vec{E} \cdot d\vec{s} = \frac{q_{enclosed}}{\epsilon_0}$$

Electrostatic field has no curl

$$\vec{\nabla} \times \vec{E} = 0$$

Laplace's equation

$$\nabla^2 V(\vec{r}) = 0$$

Poisson's equation

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

Electromagnetic energy

$$\boxed{U_E = \frac{\epsilon_0}{2} \int \vec{E}^2\, d^3r + \frac{1}{2\mu_0} \int \vec{B}^2\, d^3r}$$

Capacitor of capacitance C

$$C = \frac{Q}{V} \;\; \& \;\; U_E = \frac{1}{2} C V^2$$

Fourier basis orthogonality relation

$$\int_0^\pi \sin(mx)\sin(nx)\, dx = \frac{\pi}{2} \delta_{mn}$$

Legendre basis orthogonality relation

$$\int_{-1}^{1} \mathrm{P}_k(x)\,\mathrm{P}_l(x)\,dx = \frac{2}{2k+1}\delta_{kl}$$

Separation of variables: general solution forms for spherical symmetry

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right)P_n(\cos\theta)$$

$$V(r,\theta) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} C_n\left(\frac{r}{R}\right)^n P_n(\cos\theta) \ \text{ for } r \le R \\ \displaystyle\sum_{n=0}^{\infty} C_n\left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta) \ \text{ for } r \ge R \end{cases}$$

Potential of an electric dipole

$$V(\vec r) = \frac{1}{4\pi\epsilon_0}\frac{\vec p\cdot\hat r}{r^2}$$ ✗

Electric dipole moment

$\vec p = \sum_{i=1}^{N} q_i \vec r_i$ & $\vec p = \int_V \rho(\vec r\,')\vec r\,' d^3r'$

$\vec p = q\vec d$ ($\vec d$ points from $-q$ to $+q$)

Torque, force, and energy for an electric dipole

$\vec\tau = \vec p\times\vec E$, $\vec F = (\vec p\cdot\vec\nabla)\vec E$, $U_{dipole} = -\vec p\cdot\vec E$

Bound charge and polarization

$\rho_b(\vec r) = -\vec\nabla\cdot\vec P(\vec r)$ and $\sigma_b(\vec r) = \vec P(\vec r)\cdot\hat n$

Electric displacement field: $\vec D = \epsilon_0\vec E + \vec P$

“Gauss’s law” for electric displacement field

$\vec\nabla\cdot\vec D = \rho_{free}$ & $\oint_S \vec D\cdot d\vec s = q_{free,enclosed}$

Boundary conditions for dielectrics

$\left(\vec{D_1} - \vec{D_2}\right)_\perp = \sigma_{free}$

$\left(\vec{D_1} - \vec{D_2}\right)_\parallel = \left(\vec{P_1} - \vec{P_2}\right)_\parallel$

Linear dielectrics

$\vec D = \epsilon\vec E$ with $\epsilon = \epsilon_0(1+\chi_e)$

$\vec\nabla\cdot\vec E = \dfrac{\rho_{free}}{\epsilon}$

$\epsilon_1\dfrac{\partial V}{\partial n} - \epsilon_2\dfrac{\partial V}{\partial n} = -\sigma_{free}$ $\hat n$ points from 2 to 1

Lorentz force law: $\vec F = q\left(\vec E + \vec v\times\vec B\right)$

Force on a line current: $\vec F = I\int d\vec l\times\vec B$

Current density & continuity equation

$\vec J = \rho\vec v$ & $\dfrac{\partial\rho}{\partial t} + \vec\nabla\cdot\vec J = 0$

Biot-Savart law

$$\vec B(\vec r) = \frac{\mu_0}{4\pi}\int_V \frac{\vec J(\vec r\,')\times(\widehat{r-r'})}{|\vec r - \vec r\,'|^2}d^3r'$$

Ampère’s law

$\vec\nabla\times\vec B = \mu_0\vec J$ & $\oint_{loop}\vec B\cdot d\vec l = \mu_0\, I_{enclosed}$

No magnetic monopoles law: $\vec\nabla\cdot\vec B = 0$

Magnetic vector potential: $\vec B = \vec\nabla\times\vec A$

Coulomb gauge definition: $\vec\nabla\cdot\vec A = 0$

Vector potential in Coulomb gauge

$\nabla^2\vec A = -\mu_0\vec J$

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\int_V \frac{\vec J(\vec r\,')}{|\vec r - \vec r\,'|}d^3r'$$

Magnetic dipole potential (vector)

$$\vec A_{dipole}(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec m\times\hat r}{r^2}$$ ✗

Magnetic moment (current I, area $\vec a$): $\vec m = I\vec a$

Torque, force, and energy for a magnetic dipole

$\vec\tau = \vec m\times\vec B$, $\vec F = \vec\nabla\left(\vec m\cdot\vec B\right)$, $U_{dipole} = -\vec m\cdot\vec B$

Bound current and magnetization

$\vec J_b = \vec\nabla\times\vec M$ and $\vec K_b = \vec M\times\hat n$

Auxiliary field: $\vec H = \frac{1}{\mu_0}\vec B - \vec M$

“Ampère’s law” for the auxiliary field

$\vec\nabla\times\vec H = \vec J_{free}$ & $\oint_{loop}\vec H\cdot d\vec l = I_{free,enclosed}$

Linear magnetic material

$\vec M = \chi_m\vec H$ & $\vec B = \mu_0(1+\chi_m)\vec H = \mu\vec H$

Magnetization — Linear

$$\boxed{\vec{M} = \chi_m \vec{H}} \quad \& \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\Rightarrow \quad \vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) = \underbrace{\mu_0\left(1+\chi_m\right)}_{\mu = \mu_0(1+\chi_m)}\vec{H}$$

= magnetic permeability of the material

$$\Rightarrow \quad \boxed{\vec{B} = \mu \vec{H}}$$

for linear magnetic materials

⚠ $$\vec{\nabla}\cdot\vec{H}_{linear} = \vec{\nabla}\cdot\left(\frac{1}{\mu}\vec{B}\right) = \frac{1}{\mu}\overset{=0}{\cancel{\vec{\nabla}\cdot\vec{B}}} + \vec{B}\cdot\underbrace{\vec{\nabla}\frac{1}{\mu}}_{=0}$$

inside a material

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{H}_{linear} \neq 0 \text{ on surfaces}$$

$\neq 0$ at surface of material

Comment: For most materials (non-magnetic) $\mu \simeq \mu_0$

↳ most materials have very little effect on the magnetic.

↳ most materials are "transparent" to the B-field (i.e. you can ignore the material)

⚠ you generally cannot ignore the presence of dielectrics in electrostatics.

Major exceptions: mu-metal, $\left.\frac{\mu}{\mu_0}\right|_{\text{mu-metal}} \sim 10^5$ or higher

used for magnetic shielding.

Ferromagnetic materials: $\left.\frac{\mu}{\mu_0}\right|_{Fe, Ni, Co} \sim 10^2 - 10^5$

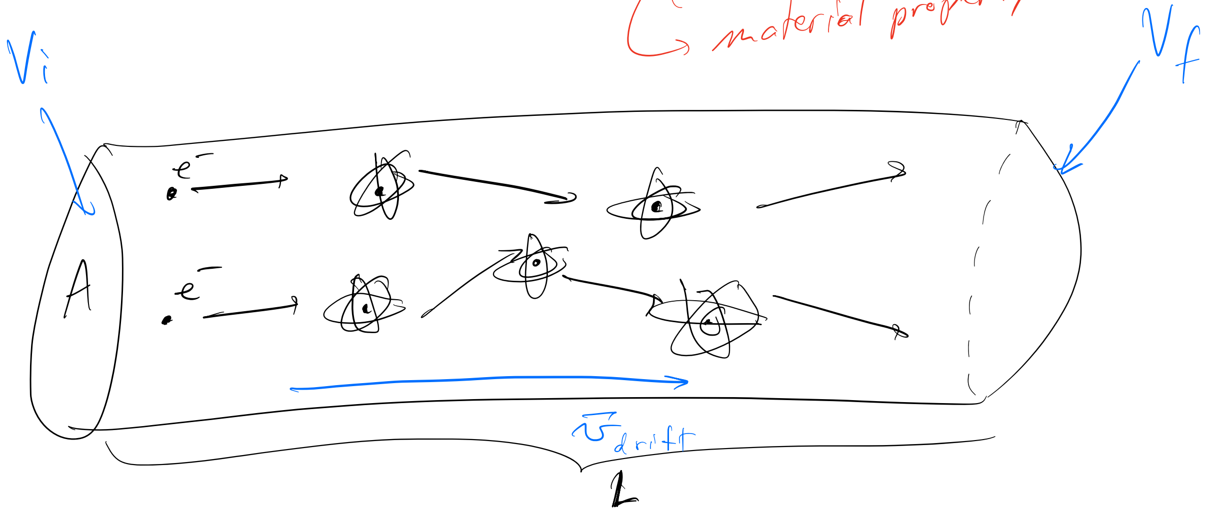
Ohm's Law: $\vec{J} = \sigma \vec{f}$

(empirical law)

$\vec{J}$: current density

σ: conductivity (constant) ↳ material property

$\vec{f}$: force on charges (per unit charge) (i.e. E)



Ohm's Law is counterintuitive, since a force should produce an acceleration, but defects/impurities in the material limit charge velocities to a drift velocity ($\lesssim m/s$)

$$\text{resistivity} = \rho = \frac{1}{\sigma}$$

Ohm's law is valid over 22 orders of magnitude!!

Examples

Conductor : $\rho_{copper} = 1.68 \times 10^{-8}\ \Omega . m$

Semiconductor : $\rho_{silicon} = 2.5 \times 10^{3}\ \Omega . m$

insulator : $\rho_{glass} = 10^{10} - 10^{14}\ \Omega . m$

Ohm's Law again: $\boxed{\vec{J} = \sigma \vec{E}}$ physicist version of Ohm's law

$$\underbrace{\int_A ds \underbrace{\int_L dz}_{L}\, \vec{J}}_{I} = \int_A ds \int_L dz\, \sigma \vec{E} = -\Delta V = V_i - V_f$$

assume constant (true for DC current)

constant

assume constant

$$\Rightarrow I L = A \underset{1/\rho}{\sigma} \underbrace{(V_i - V_f)}_{V} \Leftrightarrow V = \underbrace{\left(\frac{\rho L}{A}\right)}_{\text{resistance } R} I$$

$\Rightarrow \boxed{V = IR}$ electrician's version of ohm's Law

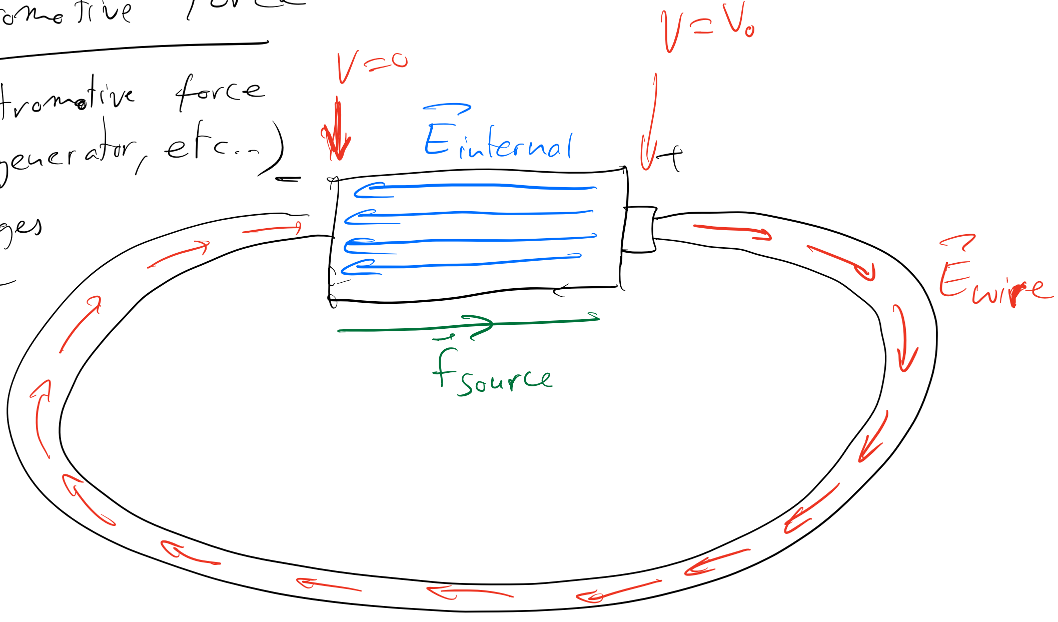
and $\boxed{R = \rho \frac{L}{A}}$

Q: If the charges travel at $\underbrace{(\sim 100\ \text{mm/s}}_{\text{drift velocity}})$, then why are electrical signals so fast?

A: All the charges move "simultaneously" under the influence of the E-field, which changes "instantaneously" (at the speed of light)

Electromotive Force

The electromotive force (battery, generator, etc..) moves charges against the internal E-field



Force per unit charge: $\vec{f}_{total} = \vec{f}_{source} + \vec{E}$

definition: Electromotive force (EMF) $= \mathcal{E} = \oint_{\substack{\text{current}\\\text{loop}}} \vec{f}_{total}\cdot d\vec{\ell}$

$$= \oint_{\substack{\text{current}\\\text{loop}}} \vec{f}_{source}\cdot d\vec{\ell} + \cancel{\oint_{\substack{\text{current}\\\text{loop}}} \vec{E}\cdot d\vec{\ell}} \rightarrow = 0$$

since $\vec{\nabla}\times\vec{E} = 0$

(Kirchoff law)

$$= \oint_{\substack{\text{current}\\\text{loop}}} \vec{f}_{source}\cdot d\vec{\ell}$$ = Energy per unit charge delivered by battery.

In wire: $\vec{f}_{source} = 0 \Rightarrow \vec{f}_{total} = \vec{E}$

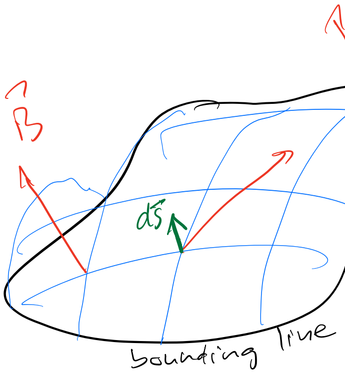
In battery: $\underbrace{\vec{f}_{source} + \vec{E} = 0}_{\text{net force on charge is zero}} = \vec{f}_{total} \Rightarrow \int_{-}^{+} (\vec{f}_{source} + \vec{E})\cdot d\vec{\ell} = 0$

$$\Rightarrow \mathcal{E} = \int_{-}^{+} \vec{f}_{source}\cdot d\vec{\ell} = \underbrace{-\int_{-}^{+} \vec{E}\cdot d\vec{\ell}}_{V_0}$$

$$\Rightarrow \boxed{\mathcal{E} = V_0}$$

Faraday's Law

definition: Magnetic Flux: $\boxed{\Phi = \int_S \vec{B}\cdot d\vec{S}}$



$d\vec{S} = ds\,\hat{n}$
$\hat{n}$: normal to surface

Universal flux rule: $\boxed{\mathcal{E}_{\text{bounding line}} = -\frac{d\Phi}{dt}}$

⚠ A changing magnetic flux induces an EMF (i.e. a voltage, not a current)

↑ generally this happens indirectly

If an E-field is responsible for $\vec{f}_{source} = \vec{E}_{source}$,

then $$\mathcal{E} = \oint \vec{f}_{total}\cdot d\vec{l} = \int \underbrace{\vec{E}_{induced}}_{\vec{E}_{source}}\cdot d\vec{l} + \underbrace{\oint \vec{E}_{wire}\cdot d\vec{l}}_{=0}$$

$$\underbrace{\neq 0}$$

then we must drop the requirement that $\vec{\nabla}\times\vec{E} = 0$ (i.e. $\vec{E} = -\vec{\nabla}V$)

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{d}{dt}\vec{B}}$$

Faraday's Law

Midterm :

Average = 67/100

High Score = 99/100