Faraday's Law : $\vec{\nabla} \times \vec{E}(t) \quad -\frac{d}{dt}\vec{B}(t)$

Magnetic flux : $\Phi = \int_{\substack{\text{surface of} \\ \text{bounding line}}} \vec{B} \cdot d\vec{s} \quad = L\,I$

Induced EMF/Voltage: $\mathcal{E}_{\substack{\text{bounding} \\ \text{line}}} = -\frac{d\Phi}{dt}$

Ohm's Law : $\vec{J} = \sigma \vec{E}$

Energy stored in an inductor L: $U_L = \frac{1}{2} L I^2$

Ampère's Improved Law : $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \underbrace{\mu_0 \varepsilon_0}_{1/c^2} \frac{\partial \vec{E}}{\partial t}$

Faraday's Law

$\vec{\nabla} \times \vec{E} = 0$ (electrostatics) $\longrightarrow$ $\boxed{\vec{\nabla} \times \vec{E} = -\frac{d}{dt} \vec{B}}$

$\Leftrightarrow -\vec{\nabla} V = \vec{E}$

(electrodynamics)

concept of a potential is gone ... until later.

⚠ Above physics does not require a current loop (i.e. a wire) to be true.

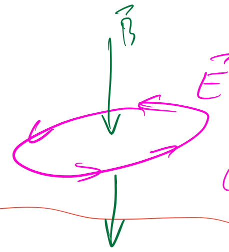
Corrolary: Lenz's law

In the presence of a current loop (i.e. a wire or conductor) an "induced" current will flow so as to minimize the change in the magnetic flux Φ ($\Phi = B \times$ area).

↳ very useful for figuring out the sign/direction of the induced current flow.

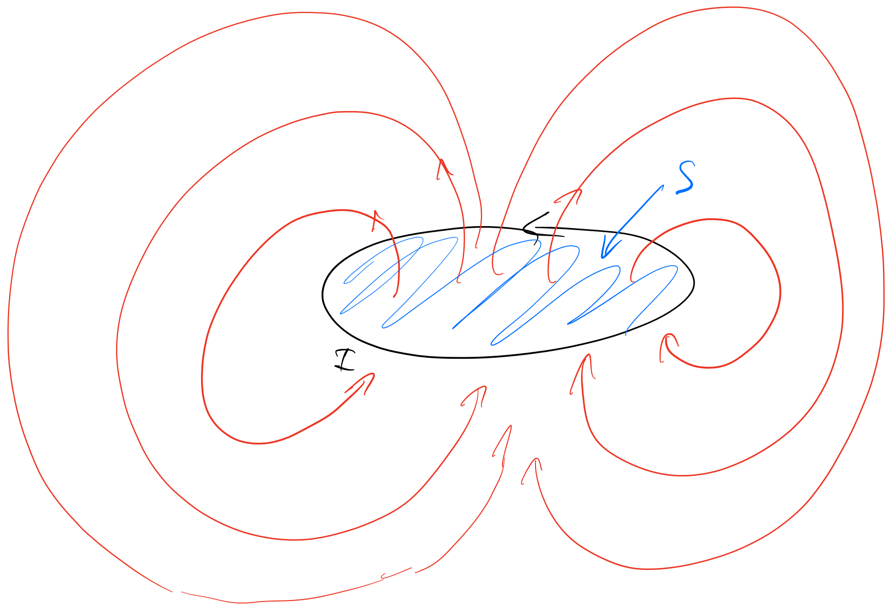
Note: Faraday's law looks like Ampère's law for an E-field in a region with no net charges.

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \end{cases} \Longleftrightarrow \begin{cases} \vec{\nabla} \cdot \vec{E} = 0 \\ \underbrace{\vec{\nabla} \times \vec{E}}_{\text{closed } \vec{E}\text{-field lines}} = \underbrace{-\frac{d\vec{B}}{dt}}_{\text{source term} \equiv \text{"current"}} \end{cases}$$



Self-inductance

A single current loop can produce a magnetic flux through itself:



recall:

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

$$= L I$$

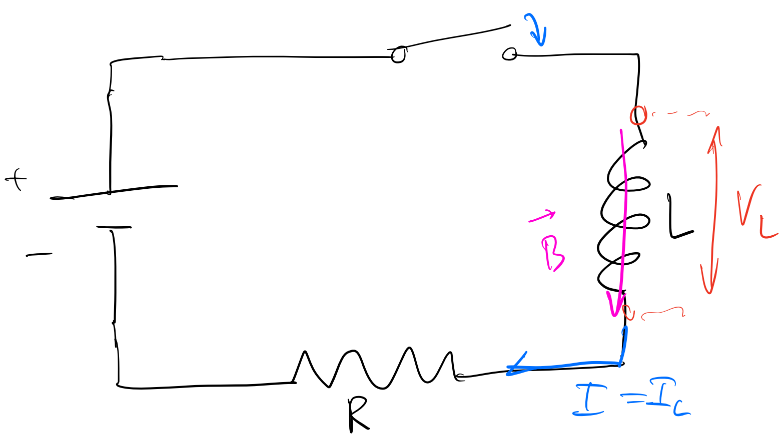
L is the self-inductance or "inductance" (measured in Henrys)

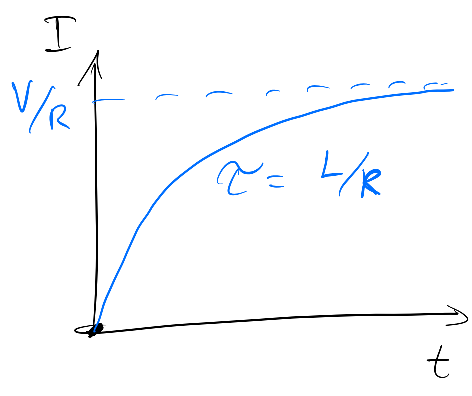
$$Z_L = i\omega L$$

note: You can compute L with $\Phi = L I = \int_S \vec{B} \cdot d\vec{s}$.

Energy stored by an inductor.

"charging up" an inductor:





EMF on the inductor : $\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(LI)$

$$\Rightarrow \boxed{\mathcal{E} = V_L = -L\frac{dI}{dt}}$$

Power : P_L = power into L = $I_L V_L = I_L\left(-L\frac{dI_L}{dt}\right)$

Energy to charge up the inductor = $U_L = \int_0^T P_L\,dt = -L\underbrace{\int_0^T I_L \frac{dI_L}{dt}\,dt}_{\frac{1}{2}I_L^2}$

$$\Rightarrow U_L = -\frac{1}{2}L\,I_L^2(T)$$

= work done to charge up the magnetic field in L.

$$\Rightarrow \boxed{U_L = \frac{1}{2}LI^2}$$ = energy stored in the B-field of L

Q: How does a B-field store energy if it cannot do work?

A: $\frac{dB}{dt} \longrightarrow \vec{E} \longrightarrow$ work done $\longrightarrow$ energy stored

$$\Rightarrow \boxed{U_{magnetic} = \frac{1}{2\mu_0}\int_{\text{all space}} \vec{B}^2\,d^3r}$$ = energy stored in a magnetic field

$$= \frac{1}{2}LI^2$$

Total Electromagnetic Energy

$$\boxed{U_{EM} = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} \vec{E}(t)^2\, d^3r + \frac{1}{2}\frac{1}{\mu_0}\int_{\text{all space}} \vec{B}(t)^2\, d^3r}$$

The problem of Ampère's law

$$\vec{\nabla}\cdot\left\{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}\right\} \Rightarrow \underbrace{\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)}_{\text{div}\cdot\text{curl}=0} = \mu_0 \underbrace{\vec{\nabla}\cdot\vec{J}}_{= -\frac{\partial\rho}{\partial t} \neq 0}$$

in electrodynamics

recall: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ conservation of charge

Ampère's law cannot be correct for time-varying $\vec{E}$ & $\vec{B}$ fields and ρ & $\vec{J}$ distributions

note: $\mu_0 \vec{\nabla}\cdot\vec{J} = -\mu_0 \frac{\partial\rho}{\partial t} = -\mu_0\varepsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{E}\right)$

insert Gauss's Law

$$= -\mu_0\varepsilon_0\, \vec{\nabla}\cdot\left(\frac{\partial}{\partial t}\vec{E}\right)$$

If we add an extra term to Ampère's law, then the problem is resolved:

Maxwell's modification of Ampère's law

$$\vec{\nabla}\cdot\left\{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right\}$$

$$\Rightarrow \quad \underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} = \underbrace{\mu_0 \vec{\nabla}\cdot\vec{J}}_{-\cancel{\mu_0 \varepsilon_0 \vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right)}} + \cancel{\varepsilon_0 \mu_0 \vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right)}$$

$$\Rightarrow \quad 0 = 0$$

<u>Ampère's Improved Law</u> : $\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$

Maxwell's Equations in Vacuum / linear matter

1) $\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \frac{\rho(\vec{r},t)}{\varepsilon_0}$ ε Gauss's Law

2) $\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$ Faraday's law

3) $\vec{\nabla}\cdot\vec{B} = 0$ $\mu_0\varepsilon_0 = \frac{1}{c^2}$ no magnetic monopoles law

4) $\vec{\nabla}\times\vec{B} = \underset{\mu}{\mu_0}\vec{J} + \underset{\mu\varepsilon}{\mu_0\varepsilon_0}\frac{\partial \vec{E}}{\partial t}$ Ampère's improved law

5) $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ conservation of charge

Displacement Current

The term " $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ " is referred to as the "displacement current".

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\Rightarrow$ Ampère's improved law : $\nabla \times B = \mu_0 J + \mu_0 J_D$

$$= \mu_0 (\vec{J} + \vec{J}_D)$$

Thinking about a time-varying $\vec{E}$-field as a "current" can be useful :

Consider a charging capacitor :