Monday, February 6, 2023

VIII Helmholtz Theorem

Short version: If you know $\vec{\nabla}\cdot\vec{F}$ and $\vec{\nabla}\times\vec{F}$, as well as the value of $\vec{F}$ on some boundary (e.g. $\vec{F}(r\to+\infty)\to 0$) then $\vec{F}$ is uniquely determined.

Long version

An arbitrary vector field $\vec{F}(\vec{r})$ can always be decomposed into a sum of 2 vector fields such that

$$\vec{F}(\vec{r}) = \vec{F}_\perp + \vec{F}_\parallel$$

with $\vec{\nabla}\cdot\vec{F}_\perp = 0$ and $\vec{\nabla}\times\vec{F}_\parallel = \vec{0}$

More precisely

$$\boxed{\vec{F}(\vec{r}) = \underbrace{\vec{\nabla}\times\vec{U}(\vec{r})}_{\vec{F}_\perp} - \underbrace{\vec{\nabla}\Omega(\vec{r})}_{\vec{F}_\parallel}}$$

So long as the integrals converge, then $\vec{U}$ and Ω are "uniquely" given by:

$$\Omega(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}_{\vec{r}'}\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \text{scalar potential}$$

$$\vec{U}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\vec{\nabla}_{\vec{r}'} \times \vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \text{vector potential}$$

<u>note</u>: $\vec{F}(\vec{r})$ must fall off faster than $\frac{1}{r}$ for $r \to +\infty$

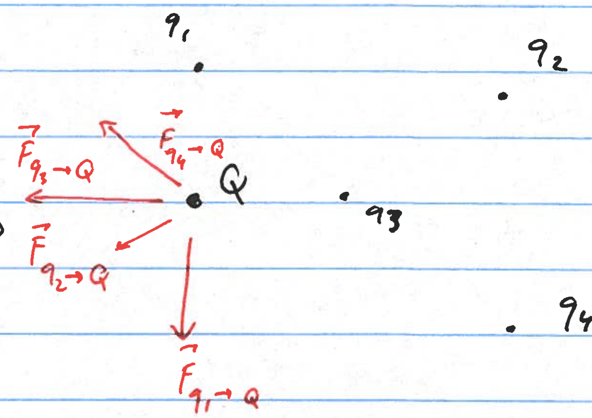
[<u>proof</u>: see PHYS 610 lecture notes (spring 2018) for March 1, 2018
└ proof uses $\nabla^2 \frac{1}{r} = -4\pi\delta^3(\vec{r})$]

<u>Corollary 1:</u> If $\vec{\nabla}\times\vec{F} = \vec{0}$, then $\vec{F}(\vec{r})$ can be derived from a <u>scalar potential</u>. → electrostatic potential V for electric field $\vec{E}$.

<u>Corollary 2:</u> If $\vec{\nabla}\cdot\vec{F} = 0$, then $\vec{F}(\vec{r})$ can be derived from a <u>vector potential</u>.
└ magnetic vector potential $\vec{A}$ for the magnetic field $\vec{B}$.

Electrostatics [chapter 2]

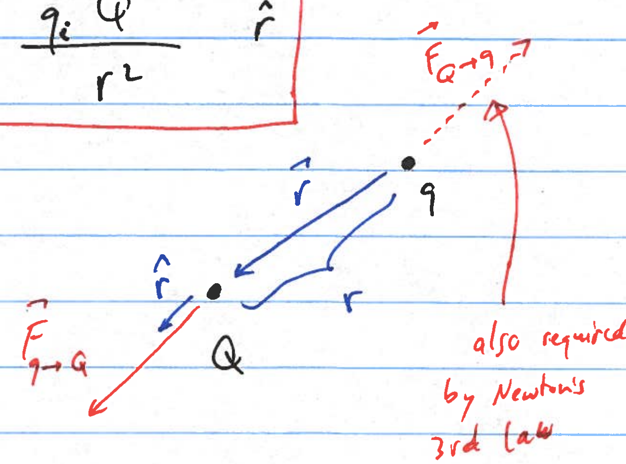
Consider a collection of stationary point charges q_1, q_2, q_3, etc and a stationary test charge Q.



Q1: what is the force on Q due to q_1 (or $q_2, q_3, \ldots$)?

A1: For stationary charges, the force is given by Coulomb's law

$$\boxed{\vec{F}_{q_i \to Q} = \frac{1}{4\pi\varepsilon_0} \frac{q_i Q}{r^2} \hat{r}}$$



ε_0 = permittivity of free space

$$= 8.85 \times 10^{-12} \underbrace{\frac{C^2}{N \cdot m^2}}_{\text{SI units}}$$

Charge Q (or q) is measured in Coulombs [SI unit].
Distance r is measured in meters [SI unit].

Q2: How strong is the Coulomb force between two 1 C charges separated by 1 m? (i.e. $q = 1C$, $Q = 1C$, $r = 1m$)

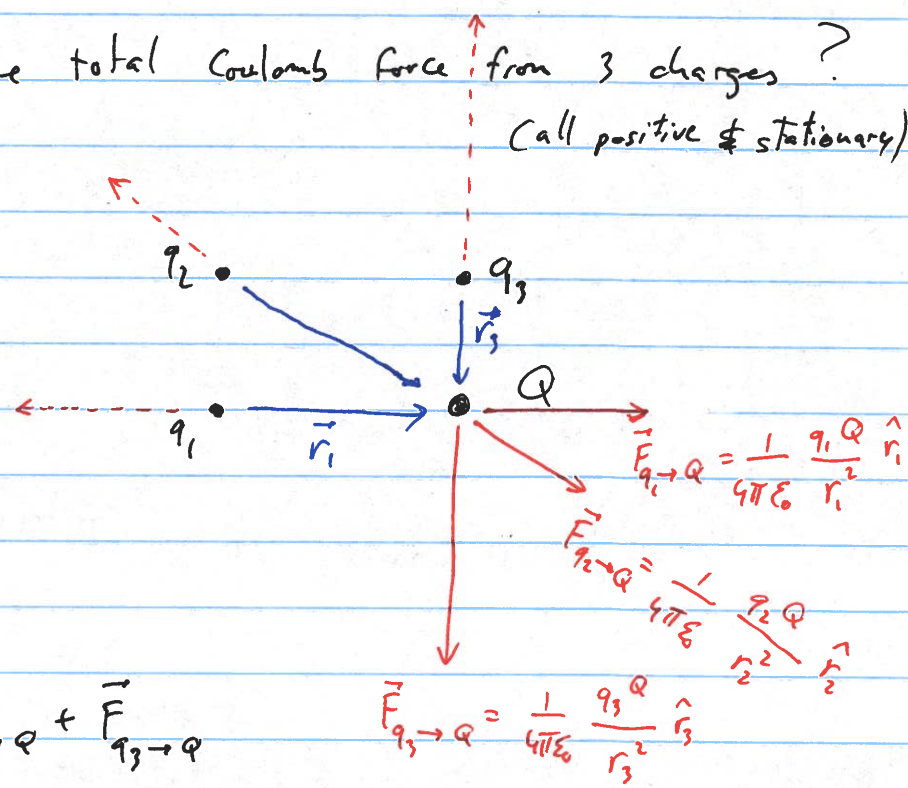
$$|\vec{F}| = \frac{1}{4\pi(8.85\times10^{-12})}\frac{(1)(1)}{(1)^2} = 8.992\times10^{9}\ N$$

$\Rightarrow$ The electric force is very strong (compared to gravity)

Note 1: 1 C is a very large charge.
$-1\ C = 6.24\times10^{18}$ electrons
$q_{e^-} = -1.602\times10^{-19}\ C$
Force between e^- separated by 5 m $\simeq$ Earth's gravity on $1 e^-$.

Note 2:
- charges of the same sign repel.
- charges of the opposite sign attract.

Q3: what is the total Coulomb force from 3 charges?
(all positive & stationary)



A3: Total Force:

$$\vec{F}_{total} = \vec{F}_{q_1\to Q} + \vec{F}_{q_2\to Q} + \vec{F}_{q_3\to Q}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{r} + \frac{q_2}{r_2^2}\hat{r}_2 + \frac{q_3}{r_3^2}\hat{r}_3\right)$$

Generalization: for N charges

$$\vec{F}_{total} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{r}_i$$

$$\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{r}_i = \vec{E}$$

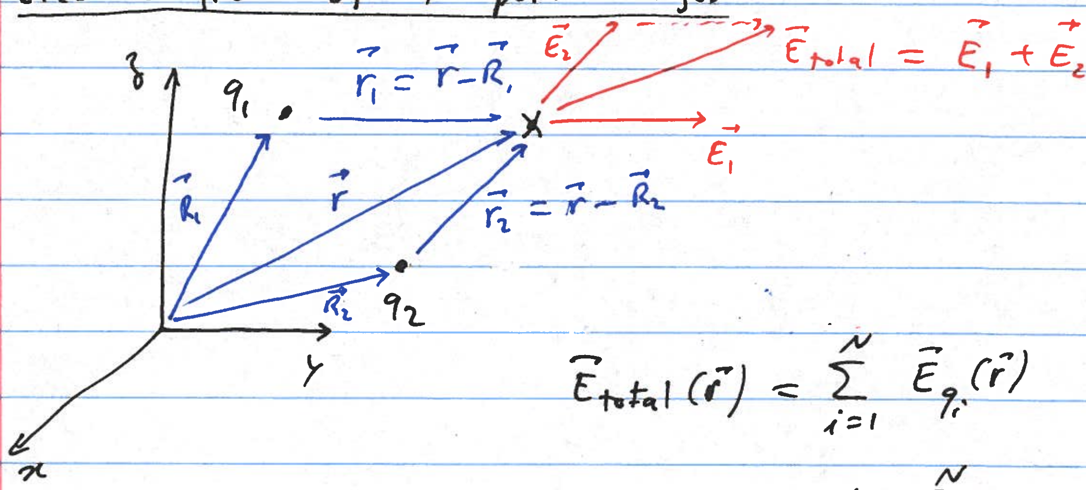
$$= Q\,\vec{E}$$

Definition: the electric field on a test charge Q is

$$\vec{E} = \frac{\vec{F}_{total}}{Q}$$

Electric field of a point charge (at origin): $\vec{E}_q(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

Electric field of N point charges

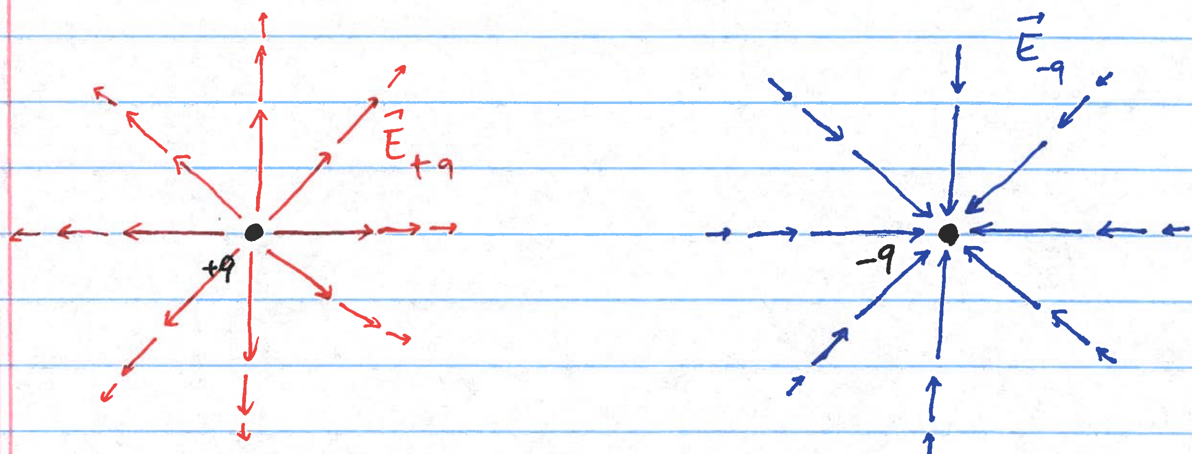


$$\vec{E}_{total}(\vec{r}) = \sum_{i=1}^{N}\vec{E}_{q_i}(\vec{r})$$

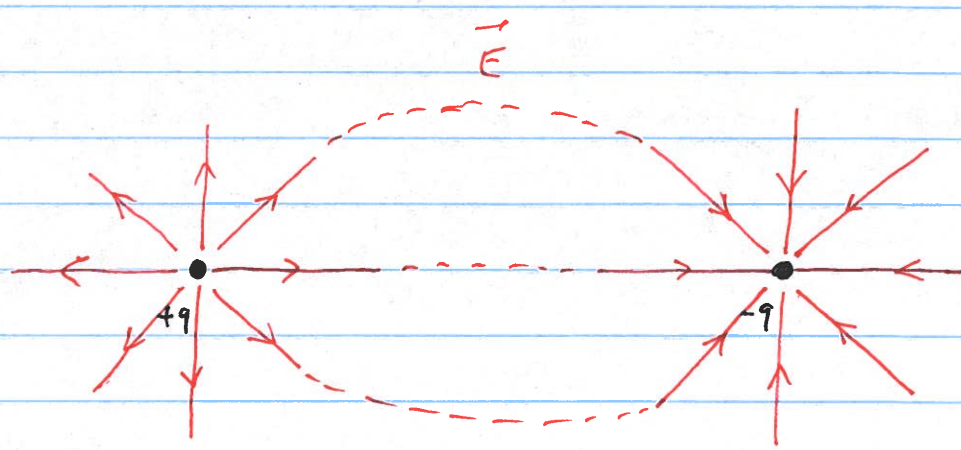
$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{R}_i|^2}\underbrace{\widehat{(\vec{r}-\vec{R}_i)}}_{\hat{r}_i}$$

Visualizing the electric field of a point charge

method 1: vector/arrow indicates direction and magnitude of $\vec{E}$.



method 2: Use continuous "field lines", which indicate the direction of the electric field. The magnitude of $\vec{E}$ is given by the density of the field lines.



note: Field lines can only end or begin on a charge.

negative charge (end) — positive charge (begin)