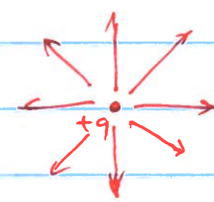
Wednesday, February 8, 2023

Pillars of Electrostatics

- Electric field of a point charge falls off like $1/r^2$

Follows naturally in a model where the $\bar{E}$-field emanates uniformily from the point charge (in 3D)



arrow density falls off as $1/r^2$

- Superposition principle : $\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 + \dots$

The electric field (i.e. Coulomb force) at a point is the sum of electric fields from all sources.

linear sum

→ See Powerpoint/PDF presentation

Continuous Charge Distributions [chpt 2.14 - 2.2.3]

Electric field: (of point charges)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{\infty} \frac{q_i}{|\vec{r}-\vec{R}_i|^2} \widehat{(\vec{r}-\vec{R}_i)}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all} \\ \text{charges}}} \left[\frac{dq}{|\vec{r}-\vec{r}\,'|^2} \widehat{(\vec{r}-\vec{r}\,')} \right]$$

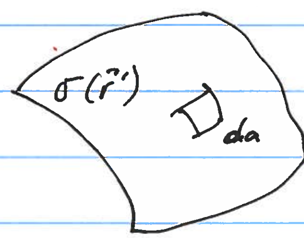
Electric field of a continuous charge distribution

$$= \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all} \\ \text{charges}}} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \, dq$$

Line charge density: $\lambda(\vec{r}\,')$ $\underset{dl'}{\text{——}}$ $dq = \lambda \, dl'$

$$dq = \underbrace{\lambda(\vec{r}\,') \, dl'}_{\text{e.g. } \lambda(x')\,dx'}$$

Surface charge density:



$$dq = \sigma(\vec{r}\,') \, da'$$

e.g. $dx'dy'$

or

cylindrical coords $\rightarrow$ $r'dr'd\phi'$, $r'd\phi'dz'$

or

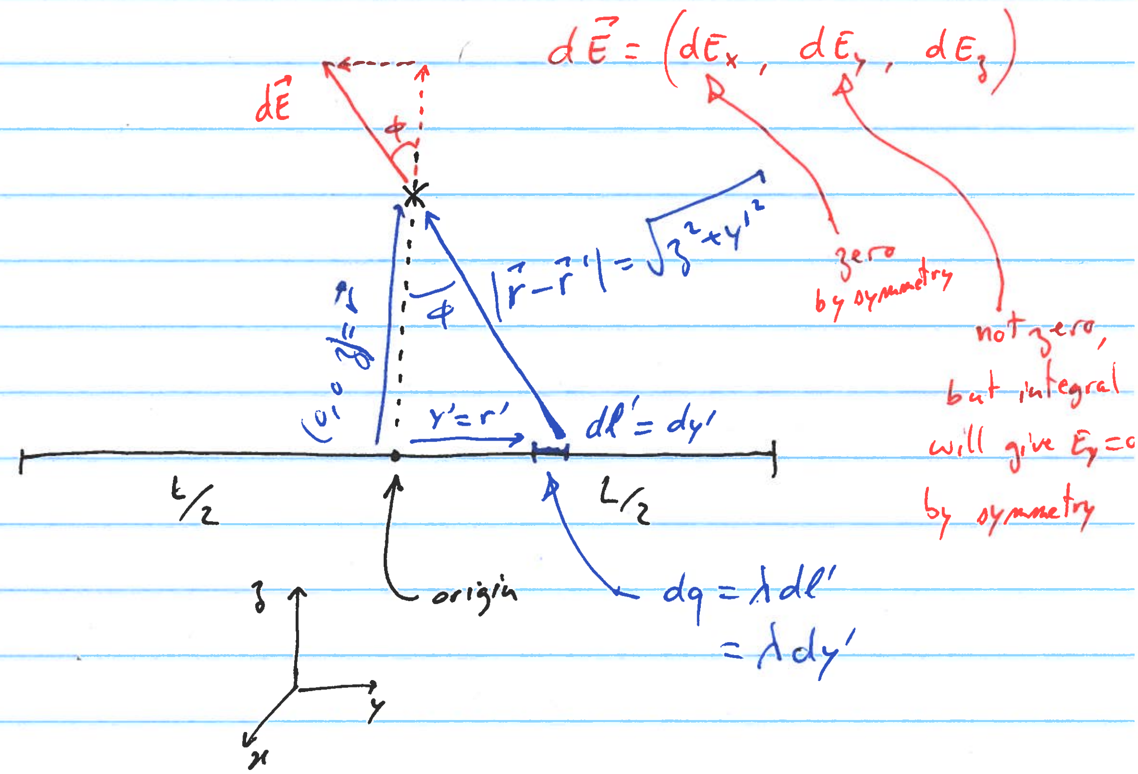
spherical coords $\rightarrow$ $r'^2 \sin\theta' \, d\phi' d\theta'$

Volume charge density:

$$dq = \rho(\vec{r}\,') \underbrace{dV'}_{\text{e.g. } dx'dy'dz'}$$

Example: Electric field from a uniform line of charge of length L and total charge q.

$\hookrightarrow \lambda = q/L =$ line charge distribution



$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy'}{\left(\sqrt{z^2+y'^2}\right)^2}\cos\phi = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy'}{\left(\sqrt{z^2+y'^2}\right)^2}\frac{z}{\sqrt{z^2+y'^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{z\lambda\, dy'}{(z^2+y'^2)^{3/2}}$$

$$\Rightarrow \vec{E} = E_z\hat{z} = \frac{1}{4\pi\varepsilon_0}\lambda z\hat{z}\int_{-L/2}^{L/2}\frac{dy'}{(z^2+y'^2)^{3/2}}$$

Apply trigonometric substitution:

$$y' = z \tan\varphi$$
$$dy' = z\sec^2\varphi \, d\varphi$$

$$= \frac{1}{4\pi\varepsilon_0} \lambda z \hat{z} \left. \frac{y'}{z^2\sqrt{z^2+y'^2}} \right|_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda \cancel{z}}{z^{\cancel{2}}} \hat{z} \underbrace{\left[\frac{L/2}{\sqrt{z^2+(L/2)^2}} - \frac{(-L/2)}{\sqrt{z^2+(-L/2)^2}} \right]}_{\frac{L}{\sqrt{z^2+(L/2)^2}}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{|z|} \frac{\lambda L}{\sqrt{z^2+(L/2)^2}} \hat{z}$$

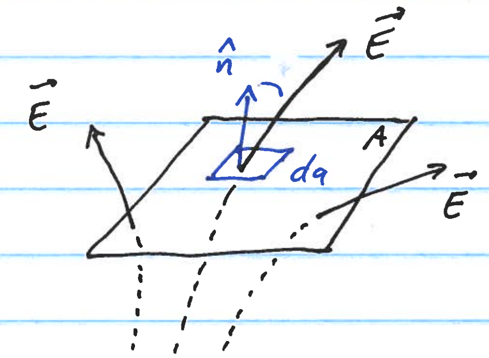
q

$$\Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{z\sqrt{z^2+(L/2)^2}} \hat{z}}$$

<u>note</u>: if $z \gg L$, then $\vec{E} \to \frac{1}{4\pi\varepsilon_0} \frac{q}{z^2}$

i.e. a line of charge behaves like a point charge if you are far enough away from it.

Electric flux = "flow" of the electric field through a surface A.

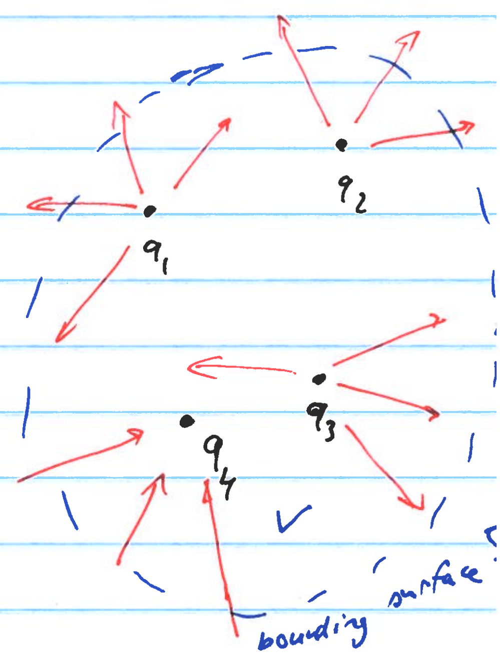


$$d\vec{a} = \hat{n}\, da \equiv d\vec{s}$$

$$\boxed{\text{electric flux (through area } A) \quad \Phi_A = \int_A \vec{E} \cdot d\vec{a}}$$

Gauss's Law

Consider a collection of charges inside a volume V with bounding surface S.



$$\vec{E}_{total}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r}-\vec{r}_i|^2} \widehat{(\vec{r}-\vec{r}_i)}$$

Electric flux through S:

$$\Phi_S = \oint_S \vec{E}_{total} \cdot d\vec{s}$$

divergence theorem

$$= \int_V (\vec{\nabla} \cdot \vec{E}_{total})\, d^3r$$

$$= \frac{1}{\cancel{4\pi}\varepsilon_0} \int_V \sum_{i=1}^{N} q_i \, \vec{\nabla}_{\vec{r}} \cdot \underbrace{\frac{\widehat{(\vec{r}-\vec{r}_i)}}{|\vec{r}-\vec{r}_i|^2}}_{\cancel{4\pi}\,\delta^3(\vec{r}-\vec{r}_i)} \, d^3r$$

$$= \frac{1}{\varepsilon_0} \sum_{i=1}^{N} q_i \underbrace{\int_V \delta^3(\vec{r}-\vec{r}_i)\, d^3r}_{=1}$$

$$= \frac{1}{\varepsilon_0} \underbrace{\sum_{i=1}^{N} q_i}_{\text{enclosed charge in } V \,=\, Q_{\text{enclosed}}}$$

$$= \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

$$\Longrightarrow \boxed{\oint_S \vec{E}_{\text{total}} \cdot d\vec{s} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}}$$

Gauss's law (integral form)

Physical Interpretation:

Each charge "emits" a fixed number of E-field lines (the number is proportional to the individual charge q_i).

Since the field lines cannot terminate in free space (or cro

then the number that cross the bounding surface give a census for the total enclosed charge.

$\underbrace{\text{number that cross the bounding surface}}_{\text{flux}}$