Monday, February 13, 2023

Gauss's law (in integral form) [chpt 2.2.1]

$$\boxed{\oint_S \vec{E}_{total} \cdot d\vec{s} = \frac{Q_{enclosed}}{\varepsilon_0}}$$

total charge enclosed in volume V (with surface S)

Physical interpretation:

Each charge "emits" a fixed number of E-field lines (the number is proportional to the individual charge q_i). Since the field lines cannot terminate in free space (or cross), then the number that cross the bounding surface give a census for the total enclosed charge.

flux

Differential form of Gauss's law [chpt. 2.2.2]

If we consider a continuous charge distribution $\rho(\vec{r})$, then

$$Q_{enclosed} = \int_V \rho(\vec{r})\, d^3r$$

plug into Gauss's law

$$\Rightarrow \underbrace{\oint_S \vec{E}_{total} \cdot d\vec{s}}_{= \int_V (\vec{\nabla} \cdot \vec{E})\, d^3r} = \frac{1}{\varepsilon_0} \int_V \rho(\vec{r})\, d^3r$$

$$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{E})\, d^3r = \int_V \frac{\rho(\vec{r})}{\varepsilon_0}\, d^3r$$

This equation holds for any volume V, including differentially small ones, so the integrands must be equal:

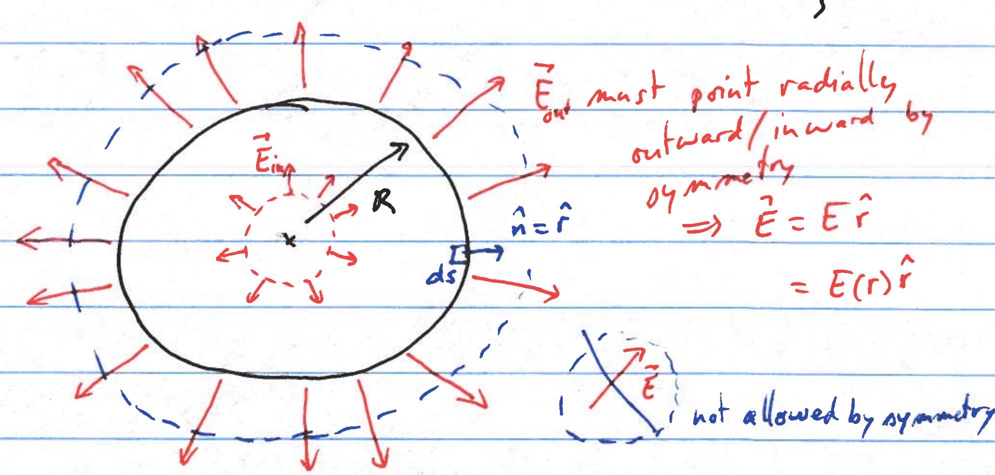
$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}}$$

Gauss's law
[Maxwell's 1st equation]

physical interpretation: you can only source or sink electric field "flow" (i.e. field-lines) with a charge ("+" = source, "−" = sink).

Applying Gauss law [chpt 2.2.3]
(useful when some symmetry is present)

Example 1: Solid sphere of uniform charge
total charge $= Q$, radius $= R$ $\Rightarrow \rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$



$\vec{E}_{out}$ must point radially outward/inward by symmetry
$\Rightarrow \vec{E} = E\hat{r}$
$= E(r)\hat{r}$

for $\underline{r > R}$: $\displaystyle \oint_S \underbrace{\vec{E}\cdot d\vec{s}}_{\substack{\vec{E} \parallel d\vec{s} \\ \Rightarrow E\,ds}} = \underbrace{\frac{Q_{enclosed}}{\varepsilon_0}}_{\frac{Q}{\varepsilon_0}}$

$$\Rightarrow \underbrace{\int_S E\,ds}_{\substack{E\underbrace{\int_S ds}_{4\pi r^2} \\ E\,4\pi r^2}} = \frac{Q}{\varepsilon_0} \Leftrightarrow E\,4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\Leftrightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}}$$ for $r > R$

i.e, E-field looks exactly like a point charge Q at the origin.

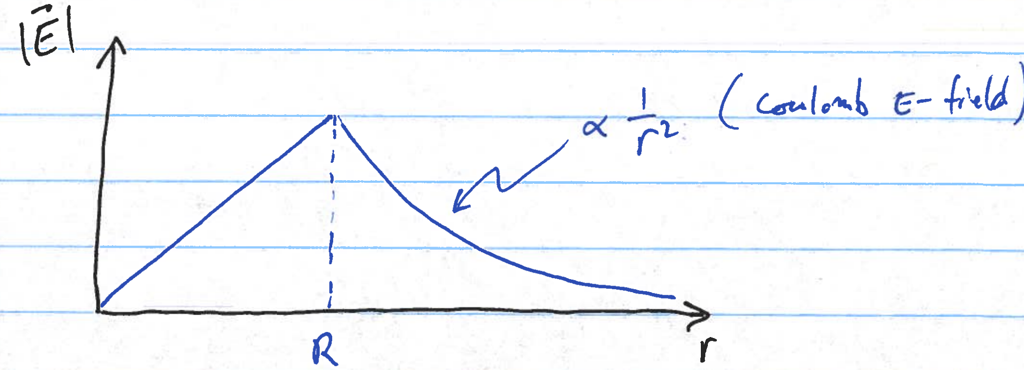
for $\underline{r < R}$: $\displaystyle \underbrace{\oint \vec{E}\cdot d\vec{s}}_{E\,4\pi r^2} = \frac{Q_{enclosed}}{\varepsilon_0} \rightarrow Q_{enclosed} = \int_V \rho\, d^3r$

$$= \frac{4}{3}\pi r^3 \rho$$

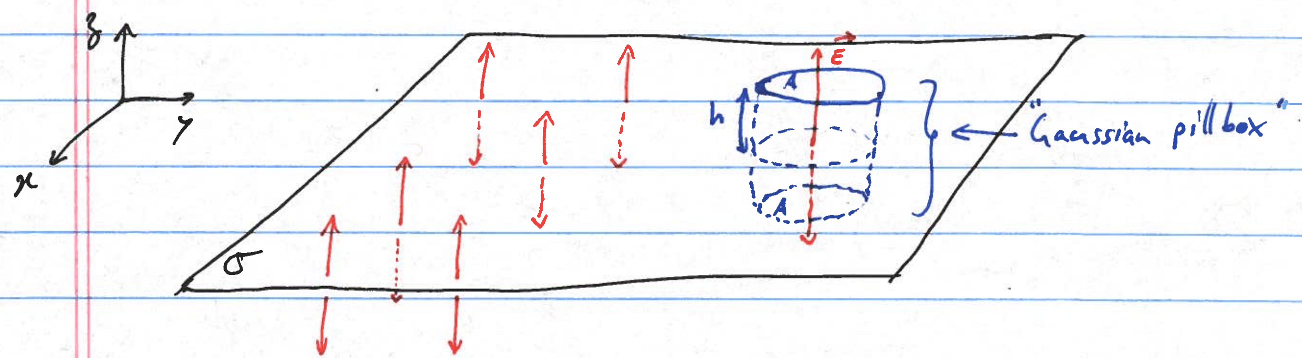
$$= \frac{\cancel{4}}{\cancel{3}}\cancel{\pi} r^3 \frac{Q}{\frac{\cancel{4}}{\cancel{3}}\cancel{\pi} R^3}$$

$$\Rightarrow E\,4\pi \cancel{r^2} = \frac{Q}{\varepsilon_0}\frac{r^{\cancel{3}}}{R^3}$$

$$\Rightarrow \boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3} r\,\hat{r}}$$ for $r < R$



Example 2: Infinite sheet of charge (in xy-plane) with uniform charge density σ.



Gauss's law:

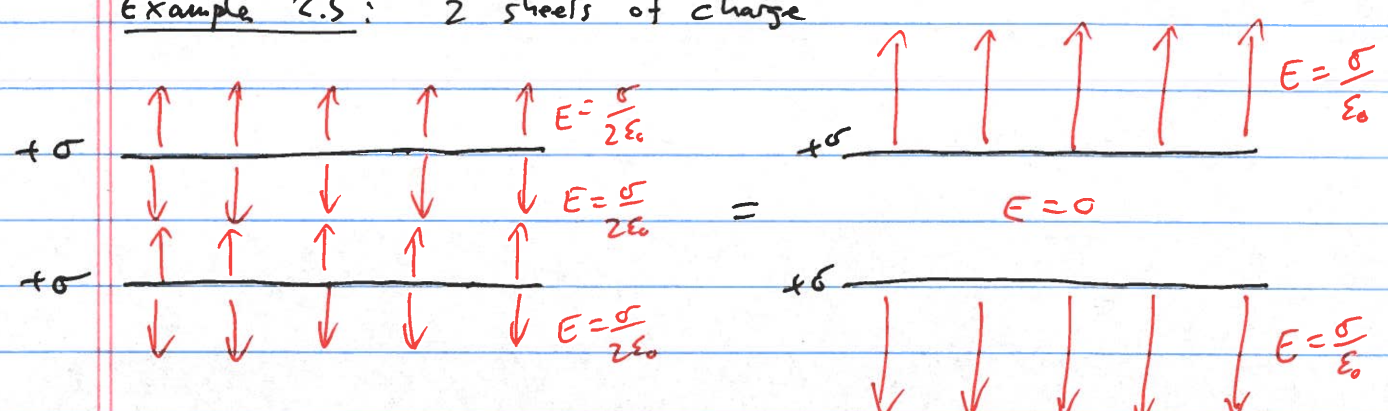
$$\underbrace{\int_{S=\text{pillbox}} \vec{E}\cdot d\vec{s}}_{\underbrace{|\vec{E}|A}_{\text{top surface}} + \underbrace{|\vec{E}|A}_{\text{bottom surface}} + \underbrace{0}_{\text{side surface (since } \vec{E}\perp\hat{n}_{\text{side}})}} = \frac{\overbrace{Q_{\text{enclosed}}}^{\sigma A}}{\varepsilon_0}$$

$$\Rightarrow \quad 2|\vec{E}|A = \frac{\sigma A}{\varepsilon_0} \quad \Leftrightarrow \quad |\vec{E}| = \frac{\sigma}{2\varepsilon_0}$$

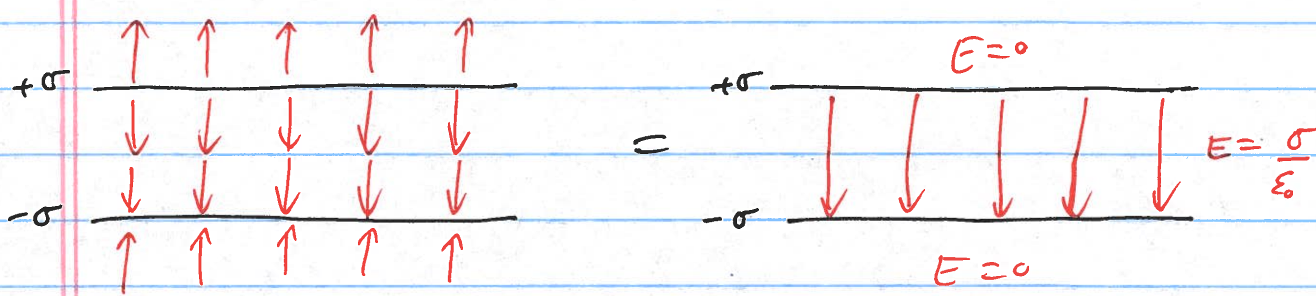
$$\Rightarrow \quad \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\,\hat{n}}$$

note: E-field is independent of distance "h" from sheet.

Example 2.5: 2 sheets of charge



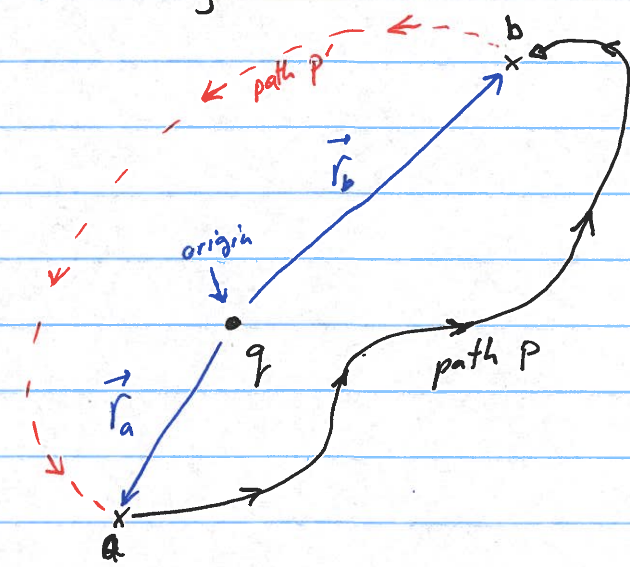
Capacitor configuration:



Curl of $\vec{E}$ [chpt 2.2.4]

Q: $\vec{\nabla}\times\vec{E} = ?$

Curl of a point charge (at origin)



Let's calculate $\displaystyle\int_{\vec{r}_a \ \text{path } P}^{\vec{r}_b} \vec{E}_q \cdot d\vec{l}$

$$\vec{E}_q = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{r}$$ (spherical coordinates)

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$ (spherical coordinates)

relative size of dr, $d\theta$, $d\phi$ depends on $d\vec{l}$ (and $\underline{r}$, θ)