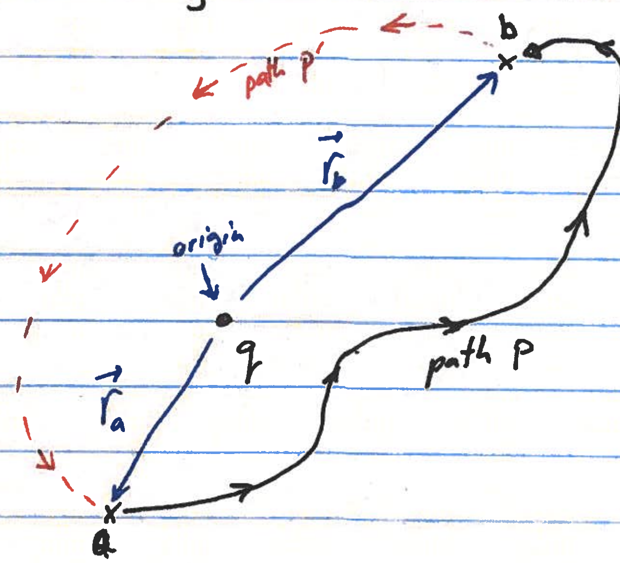
Wednesday, February 15, 2023

Curl of $\vec{E}$ [chpt 2.2.4]

Q: $\vec{\nabla}\times\vec{E} = ?$



Curl of a point charge
(at origin)

Let's calculate $\displaystyle\int_{\vec{r}_a,\ \text{path P}}^{\vec{r}_b} \vec{E}_q\cdot d\vec{\ell}$

$$\vec{E}_q = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{r} \qquad \text{(spherical coordinates)}$$

$$d\vec{\ell} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \qquad \text{(spherical coordinates)}$$

relative size of dr, $d\theta$, $d\phi$ depends on $d\vec{\ell}$ (and r, θ)

but $\vec{E}_q \cdot d\vec{l} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r^2}\, dr \quad [\hat{r}\cdot\hat{\theta} = 0,\ \hat{r}\cdot\hat{\phi} = 0]$

thus $$\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}_q \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\, q \int_{r_a}^{r_b} \frac{1}{r^2}\, dr = \frac{1}{4\pi\varepsilon_0}\, q \left(-\frac{1}{r}\right)_{r_a}^{r_b}$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$\Rightarrow$ this integral depends only on the endpoints, not the path!!

Thus $$\oint_{\text{path } p+p'} \vec{E}_q \cdot d\vec{l} = \underbrace{\int_{\vec{r}_a}^{\vec{r}_b}}_{\text{path } p} \vec{E}_q \cdot d\vec{l} + \underbrace{\int_{\vec{r}_b}^{\vec{r}_a}}_{\text{path } p'} \vec{E}_q \cdot d\vec{l} = 0$$

result does not depend on p & p'
only the endpoints

$$\boxed{\Rightarrow \oint_{\substack{\text{any}\\ \text{closed path}\\ \text{loop}}} \vec{E}_q \cdot d\vec{l} = 0} \iff \int_{\substack{\text{any surface}\\ \text{(with any bounding line)}}} (\vec{\nabla} \times \vec{E}_q) \cdot d\vec{s} = 0$$

↳ apply Stokes's theorem

Since the result is true for any surface, even an infinitesimal one, then $\Rightarrow \vec{\nabla} \times \vec{E}_q = \vec{0}$

Since any electrostatic E-field that we can have can be built by adding up the contributions of point charges, then by the <u>superposition principle</u> we have

$$\boxed{\vec{\nabla} \times \vec{E} = \vec{0}}$$

(true for any static electric field)

$$\Longleftrightarrow \quad \int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{\ell} \quad \text{is independent of path}$$

Electric Potential [chpt 2.3]

<u>Definition</u>: The potential $V(\vec{r})$ with respect to a reference point $\vec{r}_0$ is given by

$$\boxed{V(\vec{r}) = -\int_{\vec{r}_0 \text{ (any path)}}^{\vec{r}} \vec{E} \cdot d\vec{\ell}}$$

[units: Volts]

<u>note</u>: Often, $\vec{r}_0$ is taken at $|\vec{r}_0| \to +\infty$ (but not always).

⚠ $V(\vec{r})$ depends on the reference point $\vec{r}_0$ ($\vec{r}_0$ determines the voltage offset)

⚠ The <u>potential difference</u> between 2 points is independent of $\vec{r}_0$.

proof: $$V(\vec{r}_b) - V(\vec{r}_a) = -\int_{\vec{r}_0}^{\vec{r}_b} \vec{E}\cdot d\vec{\ell} - \left[-\int_{\vec{r}_0}^{\vec{r}_a} \vec{E}\cdot d\vec{\ell} \right]$$

$$= -\left\{ \int_{\vec{r}_a}^{\vec{r}_0} \vec{E}\cdot d\vec{\ell} + \int_{\vec{r}_0}^{\vec{r}_b} \vec{E}\cdot d\vec{\ell} \right\}$$

$$= -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{\ell}$$

Connection betwee $\vec{E}$ & V:

$$-\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{\ell} = V(\vec{r}_b) - V(\vec{r}_a) = \int_{\vec{r}_a}^{\vec{r}_b} (\vec{\nabla} V)\cdot d\vec{\ell}$$ Gradient Theorem

since this equality holds for all $\vec{r}_a$ & $\vec{r}_b$, then

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

note 1: - If you have $\vec{E}$, then get V with $V(\vec{r}) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}\cdot d\vec{\ell}$ (pick a path!)

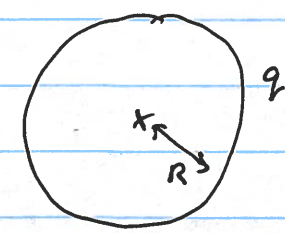
- If you have V, then get $\vec{E}$ with $\vec{E} = -\vec{\nabla} V$

note 2: V obeys the superposition principle: $V_{total} = V_{q_1} + V_{q_2} + \cdots$

note 3: In practice, it's often more efficient to calculate $V(\vec{r})$ directly, since it only has one component.
[$\vec{E} = (E_x, E_y, E_z)$ has 3 components]

Example: Potential of a uniformily charged spherical shell of radius R and charge q.

$$\hookrightarrow \quad \sigma = \frac{q}{4\pi R^2}$$

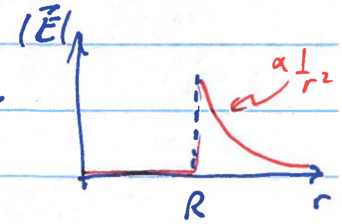


Gauss's law:

$r > R$: $\quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{r}$

$r < R$: $\quad \vec{E} = 0 \qquad$ (enclosed charge is zero)

$\Rightarrow$ $\vec{E}$ is discontinuous at shell boundary



reference point: $\vec{r}_0$ is at infinity $(|\vec{r}_0| \to +\infty)$

$$V(\vec{r}) = -\underbrace{\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}}_{\text{some path}} = -\int_{\infty}^{r} E\,dr'$$

Choose a purely radial path

$$= -\int_{\infty}^{r>R} \frac{1}{4\pi\varepsilon_0}\, q\, \frac{dr'}{r'^2} = \frac{1}{4\pi\varepsilon_0}\, q\, \frac{1}{r'}\Bigg|_{\infty}^{r>R} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$\Rightarrow \quad V_{r>R}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad \text{(also the potential for a point charge!)}$$

for $r < R$, then we have

$$V(\vec{r}) = -\int_{\infty}^{R} \vec{E}\cdot d\vec{l} - \int_{R}^{r} \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} - \overbrace{\int_{R}^{r} \vec{0}\cdot d\vec{r}'}^{=0}$$

$$\Rightarrow \boxed{\begin{aligned} V_{r\geq R}(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \\ V_{r\leq R}(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \end{aligned}}$$

note: $V(\vec{r})$ is continuous across shell boundary.

Poisson's Equation [chpt 2.3.3]

Since $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho(\vec{r})}{\varepsilon_0}$ and $\vec{E} = -\vec{\nabla} V$, then

$$\Rightarrow \vec{\nabla}\cdot(-\vec{\nabla}\cdot V) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\Leftrightarrow \boxed{\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}}$$

Poisson's equation

If no charges are present in a given volume, then

$$\boxed{\nabla^2 V(\vec{r}) = 0}$$

Laplace's equation

note 1: $\vec{\nabla}\times\vec{E} = \vec{0}$ does not give any additional information, since $\vec{\nabla}\times(-\vec{\nabla}V) = \vec{0}$ [curl-grad is always zero]

note 2: The reference point is unspecified for these equations.

↳ the reference point is implicitly defined by the boundary conditions.

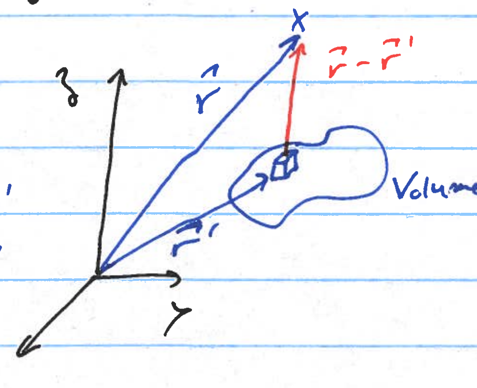
→ i.e. where $V=0$ (i.e. the "ground").

For a point charge: $V(\vec{r}) = \dfrac{1}{4\pi \varepsilon_0} \dfrac{q}{r}$

arbitrary volume of charge $\Rightarrow V(\vec{r}) = \dfrac{1}{4\pi \varepsilon_0} \displaystyle\int_{\text{Volume}} \dfrac{dq(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$

$\lambda(\vec{r}\,')\,dr'$ or $\sigma(\vec{r}\,')\,d^2r'$

$$\Rightarrow \boxed{V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \int_{\text{Volume}} \frac{\rho(\vec{r}\,')\, d^3r'}{|\vec{r} - \vec{r}\,'|}}$$



for an arbitrary charge distribution (very useful)

⚠ Assumes that the reference point is at $|\vec{r}_0| \to +\infty$

⚠ Only works if you know $\rho(\vec{r}\,')$!!!