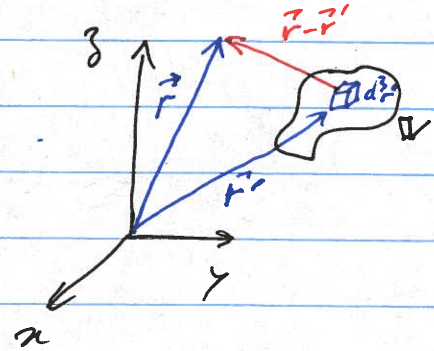
Monday, February 20, 2023

Potential from an arbitrary charge distribution

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathbb{V}\,\text{volume}} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'}$$

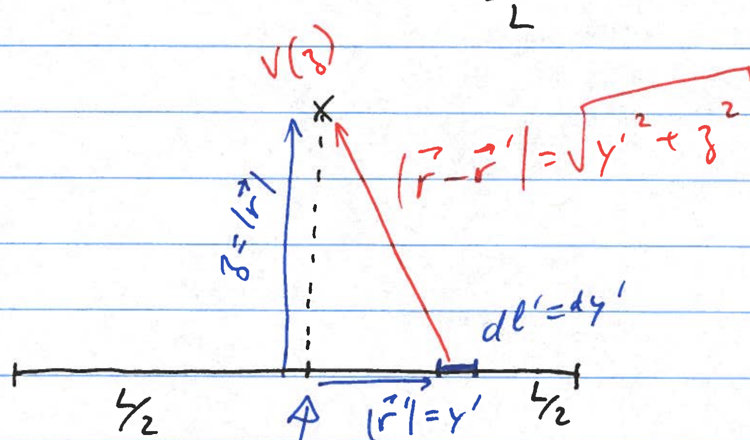


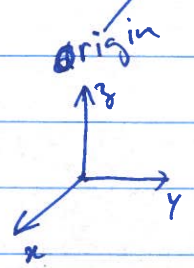
$$\rho(\vec{r}\,')\,d^3r' \rightarrow \lambda(r')\,d\ell'$$
$$\sigma(\vec{r}\,')\,d^2r'$$

⚠ assumes that reference is at $|\vec{r}_0| \rightarrow +\infty$.

Example: line of charge of length L and charge q.

L, $\lambda = \dfrac{q}{L}$





$$V(\vec{r}) = V(0,0,z)$$
$$= \frac{1}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{\lambda\, d\ell'}{\sqrt{z^2 + y'^2}}$$

$$\Rightarrow \quad V(0,0,z) = V(z) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{dy'}{\sqrt{z^2 + y'^2}}$$

lots of

[trigonometric substitution $\begin{cases} y' = z \tan\varphi \\ dy' = z \sec^2\varphi \, d\varphi \end{cases}$]

$$= \frac{\lambda}{4\pi\varepsilon_0} \frac{1}{2} \left[\ln\left(1 + \frac{y'}{\sqrt{z^2+y'^2}}\right) - \ln\left(1 - \frac{y'}{\sqrt{z^2+y'^2}}\right) \right]_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0} \cancel{\frac{1}{2}} \frac{q}{L} \left\{ \ln\left(1 + \frac{L/2}{\sqrt{z^2+(L/2)^2}}\right) - \ln\left(1 - \frac{L/2}{\sqrt{z^2+(L/2)^2}}\right) \right.$$

$$\left. \cancel{- \ln\left(1 - \frac{L/2}{\sqrt{z^2+(L/2)^2}}\right)} + \cancel{\ln\left(1 + \frac{L/2}{\sqrt{z^2+(L/2)^2}}\right)} \right\}$$

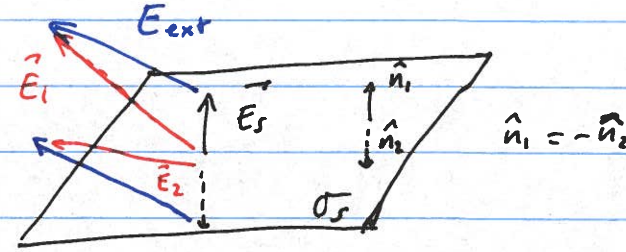
$$\Rightarrow \quad \boxed{V(0,0,z) = \frac{1}{4\pi\varepsilon_0} \frac{q}{L} \left[\ln\left(1 + \frac{L/2}{\sqrt{z^2+(L/2)^2}}\right) - \ln\left(1 + \frac{L/2}{\sqrt{z^2+(L/2)^2}}\right) \right]}$$

note: $E_z = -\frac{\partial}{\partial z} V(0,0,z) = \frac{1}{4\pi\varepsilon_0} \frac{q}{z\sqrt{z^2+(L/2)^2}}$

page of algebra

$\vec{E}$ & V at a charged surface

$$\vec{E}_s = \frac{\sigma_s}{2\varepsilon_0}\hat{n}$$



If we apply an external E-field $\vec{E}_{ext}$, then

$$\vec{E}_{total} = \vec{E}_{ext} + \vec{E}_s \Longrightarrow \vec{E}_1 = \vec{E}_{ext} + \frac{\sigma_s}{2\varepsilon_0}\hat{n}_1$$

$$\vec{E}_2 = \vec{E}_{ext} + \frac{\sigma_s}{2\varepsilon_0}\vec{n}_2 = \vec{E}_{ext} - \frac{\sigma_s}{2\varepsilon_0}\hat{n}_1$$

$$\Rightarrow \vec{E}_1 - \vec{E}_2 = \boxed{\Delta\vec{E}\Big|_s = \frac{\sigma_s}{\varepsilon_0}\hat{n}_1}$$

$$\Longrightarrow \boxed{\begin{cases} E_{1\perp} - E_{2\perp} = \dfrac{\sigma_s}{\varepsilon_0} \\ E_{1\parallel} - E_{2\parallel} = 0 \end{cases}}$$

$\Rightarrow$ normal component is discontinuous

$\Rightarrow$ parallel component is continuous

potential: $V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{\ell}$ (path P) $\Longrightarrow$ V is from an integral

$\quad\Rightarrow$ V is continuous across any surface

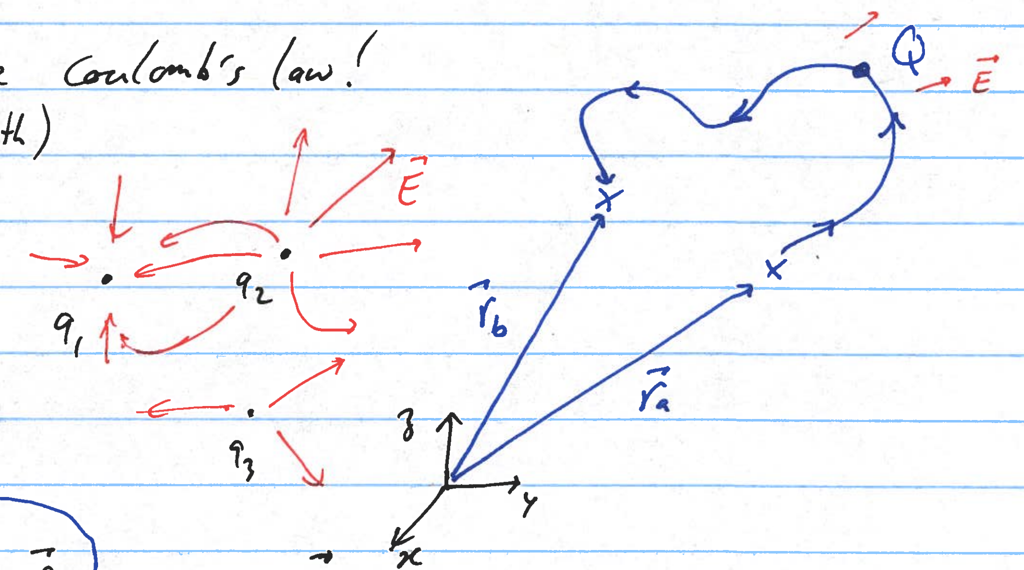
note: $\dfrac{\partial V}{\partial n} = -\dfrac{\sigma}{\varepsilon_0}$

$\dfrac{\partial V}{\partial n} = (\vec{\nabla} V)\cdot\hat{n}$

Energy

Q: Given the electric field $\vec{E}(\vec{r})$ generated by some charge distribution, how much <u>work</u> must be done to move a test charge Q from $\vec{r}_a$ to $\vec{r}_b$?
(along some path)

A: Integrate Coulomb's law!
(along path)



force provided by you (counter to Coulomb force)

$$\text{Work} = W = \int_{\vec{r}_a\,(\text{path})}^{\vec{r}_b} \vec{F}\cdot d\vec{l} = -\int_{\vec{r}_a\,(\text{path})}^{\vec{r}_b} \vec{F}_{\text{coulomb}}\cdot d\vec{l}$$

$$= -\int_{\vec{r}_a}^{\vec{r}_b} Q\,\vec{E}(\vec{r})\cdot d\vec{l} = Q\left[-\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r})\,d\vec{l}\right]$$

$$\Rightarrow \boxed{W = Q\left[V(\vec{r}_b) - V(\vec{r}_a)\right]}$$

(independent of path)
↳ potential is conservative

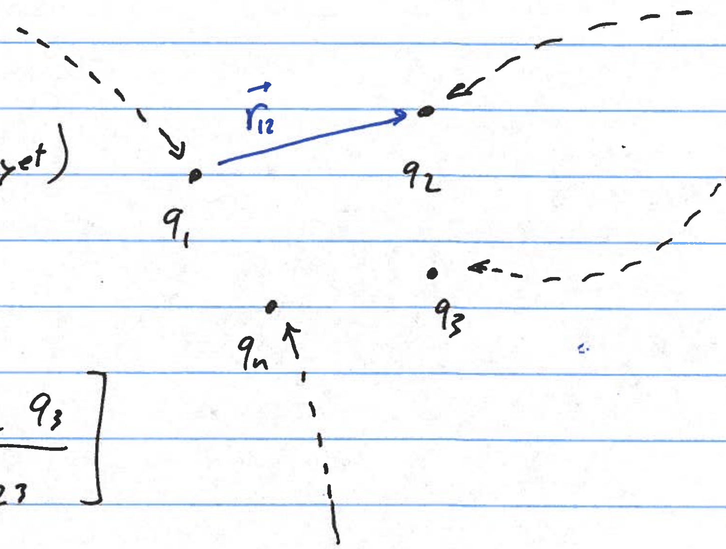
$\Rightarrow$ The potential is the <u>work</u> per unit charge.

<u>note:</u> if $|\vec{r}_a| \to +\infty$ (with $V(r\to\infty)\to 0$), then $\boxed{W = QV(\vec{r})}$

Energy of a charge distribution

(i.e. energy to assemble a charge distribution)

point charges:



work to add q_1: $W_1 = 0$ (no other charges yet)

work to add q_2: $W_2 = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q_1 q_2}{r_{12}}$

Work to add q_3: $W_3 = \dfrac{1}{4\pi\varepsilon_0} \left[\dfrac{q_1 q_3}{r_{13}} + \dfrac{q_2 q_3}{r_{23}} \right]$

$\Rightarrow$ total work to assemble all 3 charges

$$W = W_1 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]$$

for n charges:

$$\text{total work} = W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{1}{2} \sum_{j\neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{ij}} \right)}_{\text{potential at } \vec{r}_i \text{ due to all charges that are not } q_i \,=\, V_i(\vec{r}_i)}$$

$r_{ij} = |\vec{r}_i - \vec{r}_j|$

$$\Rightarrow W = \frac{1}{2} \sum_{i=1}^{n} q_i V_i(\vec{r}_i)$$

⚠ now we are counting the contribution to the potential from charge q_i.

Continuous charge distribution (generalization)

$$W = \frac{1}{2} \int_{V} \rho(\vec{r}\,') V(\vec{r}\,') \, d^3r'$$

V: volume of charge density (or bigger) however $\rho(\vec{r}') = \varepsilon_0 \vec{\nabla} \cdot \vec{E}(\vec{r}')$

So, $$W = \frac{1}{2} \int_{\mathcal{V}} \varepsilon_0 \underbrace{(\vec{\nabla} \cdot \vec{E}) V}_{\vec{\nabla} \cdot (V\vec{E}) - \vec{E} \cdot \vec{\nabla} V} \, d^3 r'$$

(from inside front cover of Griffiths "eq. 5")

$$= \frac{\varepsilon_0}{2} \left\{ \underbrace{\int_{\mathcal{V}} (\vec{\nabla} \cdot V\vec{E}) \, d^3 r'}_{\text{divergence theorem}} + \int \vec{E} \cdot (\underbrace{-\vec{\nabla} V}_{\vec{E}}) \, d^3 r' \right\}$$

$$\int_S V\vec{E} \cdot d\vec{s}$$

$$= \frac{\varepsilon_0}{2} \left\{ \int_S V \vec{E} \cdot d\vec{s} + \int_{\mathcal{V}} \vec{E}^2 \, d^3 r' \right\}$$

$V \sim \frac{1}{r'}$, $\vec{E} \sim \frac{1}{r'^2}$, $d\vec{s} \sim r'^2$

$$\underbrace{\underbrace{\frac{1}{r'} \cdot \frac{1}{r'^2} \cdot r'^2}_{\propto \frac{r'^2}{r'^3} = \frac{1}{r'}}}_{= 0 \text{ for } r' \to +\infty}$$

for $\mathcal{V} \to$ very large

$\hookrightarrow \vec{E}(r' \to +\infty) \propto \frac{1}{r'^2}$

$V(r' \to +\infty) \propto \frac{1}{r'}$

$$\Rightarrow \boxed{W = \frac{\varepsilon_0}{2} \int_{\text{all space}} \vec{E}^2(\vec{r}') \, d^3 r'}$$

always positive

$\hookrightarrow$ includes work needed to "make" point charges.

$\hookrightarrow$ this is the energy stored in the electric field.

$\Rightarrow$ energy density : $\boxed{\mathcal{E}(\vec{r}) = \frac{\varepsilon_0}{2} \vec{E}^2(\vec{r})}$