PHYS 401: Electricity & Magnetism I
Due date: Wednesday, March 1, 2023

Problem set #5

1) Problem 2.28

2) Problem 2.52

3) Problem 2.60
Answer: $\frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$

4) Mean value theorem for electrostatics
Consider a function $f(\vec{r})$ that obeys Laplace's equation $\vec{\nabla}^2 f = 0$. Show that $f(\vec{r})$ obeys the following average rule: The value of $f(\vec{r})$ at any point $\vec{r}$ is equal to the average of $f(\vec{r})$ over the surface of any sphere centered on $\vec{r}$.

Note: this result shows that $f(\vec{r})$ can have no local maximum or minimum, only saddle points at most.

5) Earnshaw's Theorem: Static Electric Field Maxima and Minima
In this problem you will prove that a charge free region of space cannot have a maximum in the magnitude of the local electric field. The theorem also applies to magnetic fields and determines the types of atoms and materials that can be trapped (levitated).

Proof by contradiction
We place the origin of our coordinate system at the position of the suspected electric field maximum. The electric field maximum at the origin is denoted as $\vec{E}(0)$. As we move away from the origin, the electric field decreases by an amount $\delta\vec{E}(\vec{r})$, so that $\vec{E}(\vec{r}) = \vec{E}(0) + \delta\vec{E}(\vec{r})$.

a) Show that the electric field must obey $\vec{E}(0)\cdot\delta\vec{E}(\vec{r}) < 0$.

b) If we choose the z-axis as the direction of the local electric maximum, then show that $\vec{E}(0)\cdot\delta\vec{E}(\vec{r}) = \vec{E}_z(0)\cdot\delta E_z(\vec{r})\hat{z}$ and $\delta\vec{E}_z(\vec{r}) < 0$.

c) Use the vector relation $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$ to show that $\nabla^2 \vec{E} = 0$, $\nabla^2 E_z = 0$, and $\nabla^2 \delta E_z = 0$.

d) Use Green's Theorem shown below to show that the average of δE_z over the surface of a sphere of radius *r* centered on the origin is equal to zero.

$$\int_V \left[\phi(\nabla^2 \psi) - \psi(\nabla^2 \phi)\right] d^3 r = \int_S \left[\phi(\nabla \psi) - \psi(\nabla \phi)\right] \cdot d\vec{s}$$

hint: use $\psi = \delta E_z$ *and* $\phi = 1/r$.

e) Show that $\vec{E}(0)$ is not an electric field maximum.

f) Give an example of a charge distribution which generates a local ***minimum*** in the electric field magnitude in a region of space free of charges. Draw a sketch of the charge distribution and the electric field minimum region.