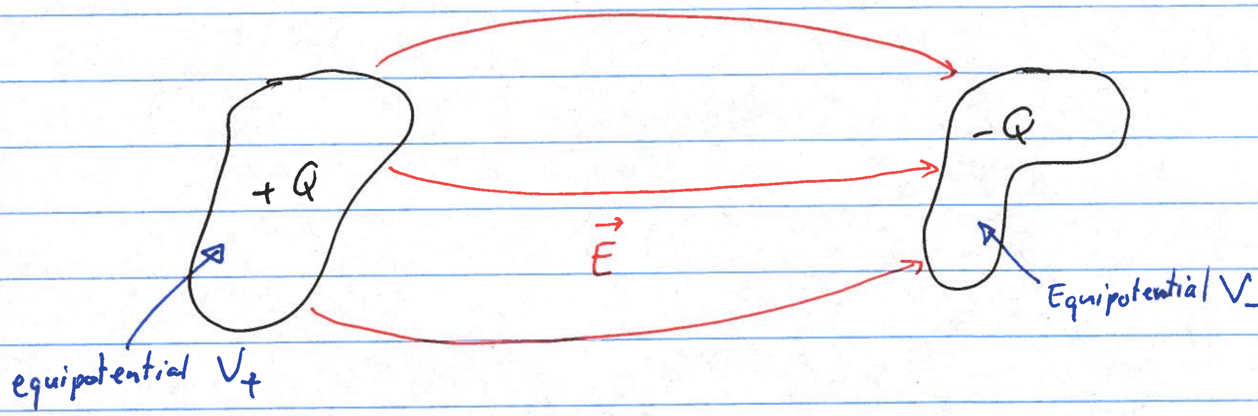
Monday, February 27, 2023

Capacitors [chpt 2.5.4]

Consider two conductors with charge $+Q$ and $-Q$.



The E-field strength is proportional to Q
The potential difference is proportional to Q
$\iff \begin{cases} \vec{E} \propto Q \\ \Delta V = (V_+ - V_-) \propto Q \end{cases}$

Definition: The capacitance C is the proportionality constant between ΔV and Q:

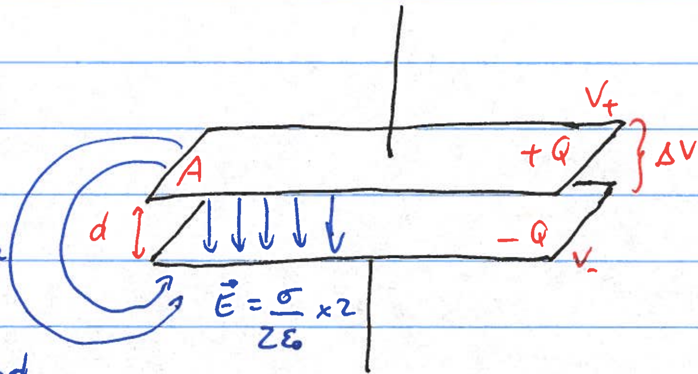
$$C = \frac{Q}{\Delta V} = [\text{Coulomb per Volt}] = [\text{Farads}]$$

$$C \, \Delta V = Q$$

note: the capacitance is a geometric quantity.

Example: Parallel plate capacitor

neglect the fringe field for $\sqrt{A} \gg d$



charges tend to gather on inner surfaces of the plates

$$\Delta V = -\int_{V_-\,\text{plate}}^{V_+\,\text{plate}} \vec{E}\cdot d\vec{\ell} = Ed \Rightarrow E = \frac{\Delta V}{d} = 2\frac{\sigma}{2\varepsilon_0}$$

σ: $\frac{Q}{A}$

note: $\Delta V = \frac{\sigma}{\varepsilon_0} d$

$$\Rightarrow \frac{\Delta V}{d} = \frac{Q/A}{\varepsilon_0} \Leftrightarrow \frac{\varepsilon_0 A}{d} \underbrace{\Delta V}_{\text{potential difference between plates}} = \underbrace{Q}_{\text{charge on a single plate}}$$

$$\Rightarrow \boxed{C = \frac{\varepsilon_0 A}{d}}$$

Energy stored in a capacitor

If you start with an uncharged capacitor, then the work to move charge from one conductor plate/terminal to the other is

$$dW = \underbrace{V}_{q/C} dq \Rightarrow W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{C} \frac{q^2}{2}\Bigg|_0^Q$$

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C \Big(\underbrace{\frac{Q}{C}}_{V}\Big)^2$$

$$\Rightarrow \boxed{W = \text{stored energy} = \frac{1}{2} C V^2} = \frac{1}{2}\frac{Q^2}{C}$$

The energy is stored in the electric field of the capacitor.

$\Rightarrow$ anything that has an electric field has a capacitance.

$\Rightarrow$ you can also calculate capacitance C with

$$\boxed{\frac{\varepsilon_0}{2}\int \vec{E}^2 d^3r = \frac{1}{2} C V^2}$$

note: $\text{Energy} = \frac{\varepsilon_0}{2}\int \vec{E}^2 d^3r = \frac{\varepsilon_0}{2}\underbrace{\left(\frac{\sigma}{\varepsilon_0}\right)^2}_{\vec{E}^2} \underbrace{Ad}_{\text{Volume}}$

$$= \frac{1}{2}\underbrace{\frac{\varepsilon_0 A}{d}}_{C}\;\underbrace{\frac{\sigma^2}{\varepsilon_0^2} d^2}_{V^2} = \frac{1}{2} C V^2$$

Calculating Electric Fields and Potentials [chpt 3]

I Laplace's Equation: Uniqueness Theorems [chpt 3.15, 3.16]

In a region of space without charges, the electric potential $V(\vec{r})$ must satisfy Laplace's equation:

$$\boxed{\nabla^2 V(\vec{r}) = 0 \iff \frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0}$$

note: solving this equation (+ boundary conditions) can be quite difficult.

Uniqueness Theorem #1 : The potential $V(\vec{r})$ in a volume $\mathbb{V}$ is uniquely determined if (a) the charge density throughout $\mathbb{V}$, and (b) the value of $V(\vec{r})$ on all boundaries, are specified.

Solution strategy:

or solution form

If you can guess a solution that satisfies $\vec{\nabla}^2 V = 0$ and the boundary conditions, then it must be the correct solution.

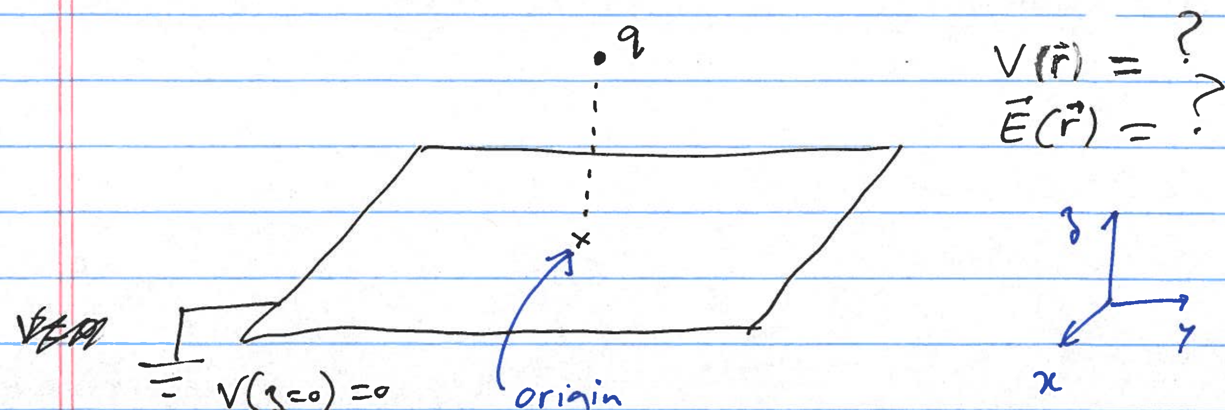
Uniqueness theorem #2: In a volume $\mathbb{V}$ surrounded by conductors (or extending to infinity) and containing a specified charge density $\rho(\vec{r}')$, then the E-field is uniquely determined if the total charge on each conductor is specified.

the charge on the conductors can move around.

II Method of Images

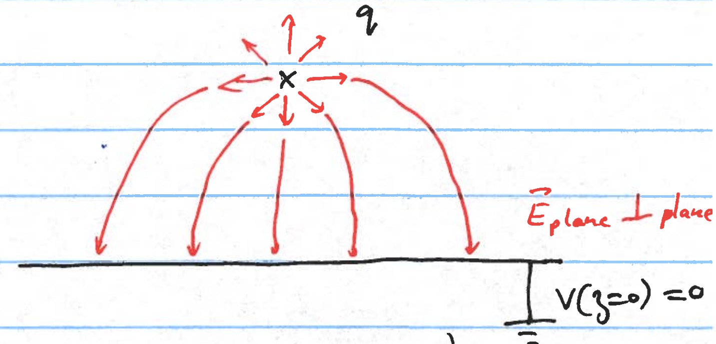
Standard Example: A point charge q is held a distance d above an infinite grounded conducting plane.

(i.e. $V_{conductor} = 0$)



Q: what are the potential $V(\vec{r})$ and E-field $\vec{E}(\vec{r})$ in the region above the plane?

qualitative guess



near the point charge: $\vec{E}_q(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0} \frac{q \,(x,y,z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}}$

$$V_q(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{x^2+y^2+(z-d)^2}}$$

The solution sort of looks like:

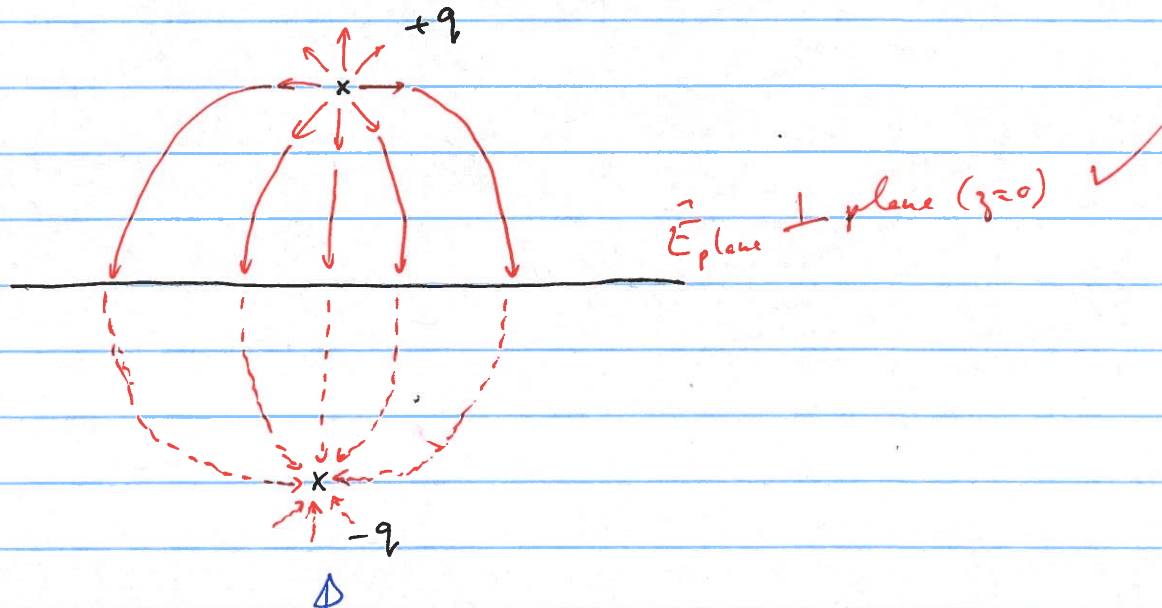


image charge

quantitative guess:

image charge (fictional)

possible solution: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$

we note: $V(z=0) = 0$ (i.e. at the conducting plane)

$V(r\to+\infty) = 0$ (for $z>0$)

$\hookrightarrow$ These are the boundary conditions and they are satisfied.

$\boxed{\Rightarrow \text{This must be the solution by the uniqueness theorem (\#1)}}$

⚠ – The solution is only valid for $z>0$.
– the image charge is fictional.

Q: What's the surface charge density?

$$\Delta\vec{E} = \vec{E}_{z>0} - \underbrace{\vec{E}_c}_{=0} = \frac{\sigma}{\varepsilon_0}\hat{n} \iff \vec{E}(z=0^+) = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\Rightarrow -\vec{\nabla}V = \frac{\sigma}{\varepsilon_0}\hat{n} \iff -\frac{\partial V}{\partial z}\bigg|_{z=0^+}\hat{z} = \frac{\sigma}{\varepsilon_0}\hat{z}$$

$$\frac{\partial V}{\partial x}\bigg|_{z=0^+} = \frac{\partial V}{\partial y}\bigg|_{z=0^+} = 0$$