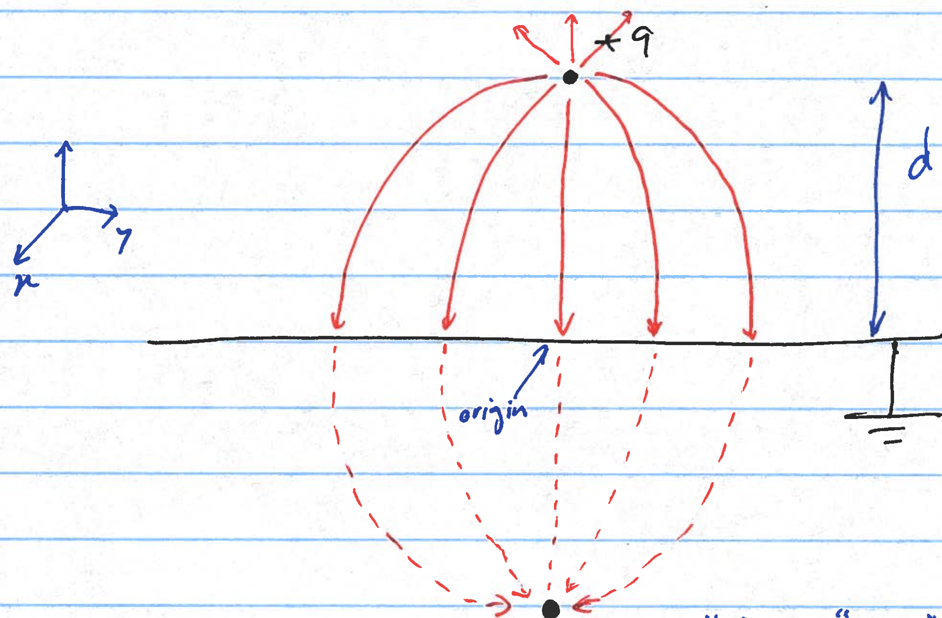
Wednesday, March 1, 2023

Method of Images (continued)

point charge above an infinite conducting grounded plane



$-q$ "image" or "mirror" charge

Solution: $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\Bigg[\underbrace{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\text{point charge potential}} - \underbrace{\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}}_{\text{image charge potential}}\Bigg]$$

↳ satisfies b.c. $V(z=0)=0$, $V(r\to\infty)=0$

the solution is only valid in the region without the image charge (i.e. $z>0$)

Q: what's the surface charge density?

$$\Delta E_{surface} = \vec{E}_{z=0^+} - \vec{E}_c = \frac{\sigma}{\varepsilon_0}\hat{n} \iff \vec{E}(z=0^+) = \frac{\overset{\sigma(x,y)}{\sigma}}{\varepsilon_0}\hat{n}$$

$$\Rightarrow -\nabla V\Big|_{z=0^+} = \frac{\sigma}{\varepsilon_0}\hat{n} \iff -\frac{\partial V}{\partial z}\Big|_{z=0^+}\hat{z} = \frac{\sigma(x,y)}{\varepsilon_0}\hat{z} \quad \left[\frac{\partial V}{\partial x}\Big|_{z=0^+}=0,\ \frac{\partial V}{\partial y}\Big|_{z=0^+}=0\right]$$

$$\left.\frac{\partial V}{\partial x}\right|_{z=0} = \frac{1}{4\pi\varepsilon_0} q \left[\left(-\frac{1}{2}\right) \frac{2x}{[x^2+y^2+(\underbrace{z}_{=0}-d)^2]^{3/2}} - \left(\frac{-1}{2}\right) \frac{2x}{[x^2+y^2+(\underbrace{z}_{=0}+d)^2]^{3/2}} \right]$$

$$= 0 \qquad \text{also} \quad \left.\frac{\partial V}{\partial y}\right|_{z=0} = 0$$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{1}{4\pi\epsilon_0} q \left[\cancel{\left(-\frac{1}{2}\right)} \frac{\cancel{2}(\cancel{z}-d)}{[x^2+y^2+(\cancel{z}-d)^2]^{3/2}} - \cancel{\left(-\frac{1}{2}\right)} \frac{\cancel{2}(\cancel{z}+d)}{[x^2+y^2+(\cancel{z}+d)^2]^{3/2}} \right]$$

$$= \frac{1}{4\pi\varepsilon_0} q \frac{2d}{[x^2+y^2+d^2]^{3/2}}$$

since $\left.\frac{\partial V}{\partial z}\right|_{z=0} = -\frac{\sigma}{\varepsilon_0} \Rightarrow \frac{\sigma}{\varepsilon_0} = -\frac{1}{4\pi\varepsilon_0} q \frac{2d}{[x^2+y^2+d^2]^{3/2}}$

$$\Rightarrow \boxed{\sigma(x,y) = -\frac{q}{2\pi} \frac{d}{[x^2+y^2+d^2]^{3/2}}}$$

induced charge density on the conducting plane

Total induced charge $= q_c = \int \sigma \, ds$

$$= -\frac{q}{2\pi} d \int_{xy \text{ plane}} \frac{dx\,dy}{[x^2+y^2+d^2]^{3/2}} = -\frac{q\,d}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{r\,dr\,d\phi}{[r^2+d^2]^{3/2}}$$

switch to polar coordinates

$$= -\frac{q}{2\pi} d \, 2\pi \int_0^\infty \frac{r\,dr}{[r^2+d^2]^{3/2}} = -q\,d \left(\frac{-1}{\sqrt{r^2+d^2}} \right)_0^\infty = -q\,d\left(0 - \left(\frac{-1}{|d|}\right)\right) = -q$$

Thus the induced charge = $\boxed{q_c = -q}$ (as expected)

Force on the charge q (by the conducting plane)

$$\vec{F} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{-q}{(2d)^2}}_{\text{Coulomb force}}\, q\,\hat{z}$$

due to image charge $\Rightarrow$ charge is attracted to the conducting plane.

Electrostatic energy is half that for the $\underbrace{\text{charge + image charge}}_{\text{half of universe is missing}}$

$$W = \int_\infty^d \vec{F}\cdot \underbrace{d\vec{l}}_{-dz} = \frac{1}{4\pi\varepsilon_0}\int_\infty^d \frac{q^2}{4z^2}\,dz$$

reference point is $r_0 \to \pm\infty$

$$= \frac{q^2}{4\pi\varepsilon_0}\frac{1}{4}\left(-\frac{1}{z}\right)\Bigg|_\infty^d = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{4}\left(-\frac{1}{d} - 0\right)$$

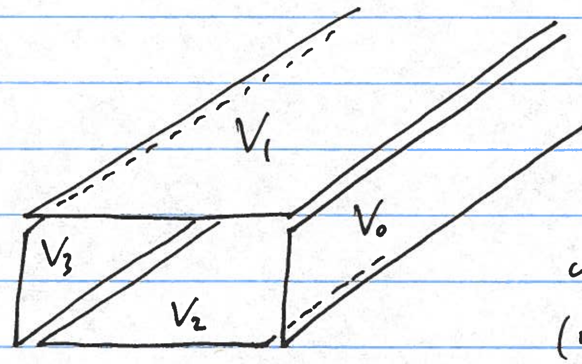
$$= -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

$$\Rightarrow \boxed{W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}}$$

Separation of Variables [chpt 3.3]

Consider a system with Cartesian Symmetry (in the geometry)

Example:



Rectangular duct with conducting walls (no extra charges)

Q: what is the potential inside the duct?

no free charges (conducting boundaries have internal charges)

↳ Laplace's equation: $\nabla^2 V = 0$ (in empty volume of duct)

$$\Leftrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x,y,z) = 0$$

Ansatz: We will consider separable solutions of the form

motivated by the symmetry of the geometry

each coordinate gets its own function

$$\Rightarrow V(x,y,z) = X(x)Y(y)Z(z)$$

$$\Rightarrow Y(y)Z(z)\frac{\partial^2}{\partial x^2}X(x) + X(x)Z(z)\frac{\partial^2}{\partial y^2}Y(y) + X(x)Y(y)\frac{\partial^2}{\partial z^2}Z = 0$$