Monday, March 6, 2023

Separation of Variable – Cartesian Symmetry
[chpt 3.3.1]

Consider a system with cartesian symmetry. In regions with no charges, the solution obey's Laplace's equation:

$$\nabla^2 V(x,y,z) = 0 \Leftrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x,y,z) = 0$$

We consider separable solutions: $V(x,y,z) = X(x)Y(y)Z(z)$

$$\Rightarrow Y(y)Z(z)\frac{\partial^2}{\partial x^2}X(x) + X(x)Z(z)\frac{\partial^2}{\partial y^2}Y(y) + X(x)Y(y)\frac{\partial^2}{\partial z^2}Z(z) = 0$$

$\hookrightarrow$ divide by $X(x)\,Y(y)\,Z(z)$

Divide by $X(x)\,Y(y)\,Z(z)$

$$\Rightarrow \frac{1}{X(x)}\frac{\partial^2}{\partial x^2}X(x) + \frac{1}{Y(y)}\frac{\partial^2}{\partial y^2}Y(y) + \frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) = 0$$

Since the solution must work for all (x,y,z) positions in the solution volume, then each term is equal to a constant such that...

$$\Rightarrow \begin{cases} \dfrac{1}{X(x)}\dfrac{\partial^2}{\partial x^2}X(x) = \underbrace{C_x}_{\alpha^2} & \Leftrightarrow \quad \boxed{\dfrac{\partial^2}{\partial x^2}X = C_x X = \alpha^2 X} \\ \dfrac{1}{Y(y)}\dfrac{\partial^2}{\partial y^2}Y(y) = \underbrace{C_y}_{\beta^2} & \Leftrightarrow \quad \boxed{\dfrac{\partial^2}{\partial y^2}Y = C_y Y = \beta^2 Y} \\ \dfrac{1}{Z(z)}\dfrac{\partial^2}{\partial z^2}Z(z) = \underbrace{C_z}_{\gamma^2} & \Leftrightarrow \quad \boxed{\dfrac{\partial^2}{\partial z^2}Z = C_z Z = \gamma^2 Z} \end{cases}$$

with $C_x + C_y + C_z = 0 \quad \Leftrightarrow \quad \boxed{\alpha^2 + \beta^2 + \gamma^2 = 0 \text{ with } \alpha, \beta, \gamma \in \mathbb{C}}$

Solutions:
$$X_\alpha = \begin{cases} A_0 + B_0 x & \alpha = 0 \\ A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0 \end{cases}$$

$$Y_\beta = \begin{cases} C_0 + D_0 y & \beta = 0 \\ C_\beta e^{\beta y} + D_\beta e^{-\beta y} & \beta \neq 0 \end{cases}$$

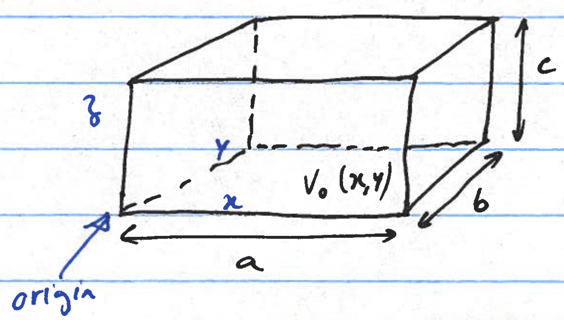
$$Z_\gamma = \begin{cases} E_0 + F_0 z & \gamma = 0 \\ E_\gamma e^{\gamma z} + F_\gamma e^{-\gamma z} & \gamma \neq 0 \end{cases}$$

General Solution: We consider any linear superposition of these solutions.

$$V(x,y,z) = \sum_{\alpha,\beta,\gamma} X_\alpha(x)\, Y_\beta(y)\, Z_\gamma(z) \underbrace{\delta\left(\alpha^2+\beta^2+\gamma^2\right)}_{\text{enforces } \alpha^2+\beta^2+\gamma^2=0}$$

Superposition coefficients are inside X_α, Y_β, Z_γ

Example: Consider a rectangular conducting box with 5 sides at potential $V=0$ and one side at potential $V_0(x,y)$.



boundary conditions:

$$\boxed{\begin{cases} X_\alpha(x=0)=0 \\ X_\alpha(x=a)=0 \end{cases} \qquad \begin{cases} Y_\beta(y=0)=0 \\ Y_\beta(y=b)=0 \end{cases} \qquad Z_\gamma(z=c)=0}$$

and $V(x,y,0) = V_0(x,y)$

harder

For X_α:

$$X_\alpha(x=0)=0 \Rightarrow A_\alpha + B_\alpha = 0 \iff A_\alpha = -B_\alpha$$

$$X_\alpha(x=a)=0 \Rightarrow A_\alpha e^{\alpha a} - A_\alpha e^{-\alpha a} = 0$$

$$\iff e^{\alpha a} = e^{-\alpha a}$$

$$\iff e^{2\alpha a} = 1 \Rightarrow 2\alpha a = 0 \Rightarrow \alpha = 0$$

or

$$2\alpha a = n i 2\pi$$

(with $n=0,\pm1,\pm2,\dots$) $\Rightarrow \boxed{\alpha = i\frac{n\pi}{a}}$

$$\Rightarrow X_\alpha(x) = 2i\,A_\alpha \left[\frac{e^{i\frac{n\pi}{a}x} - e^{-i\frac{n\pi}{a}x}}{2i}\right]$$

$$\Rightarrow \boxed{X_n(x) = A_n \sin\left(\frac{n\pi}{a}x\right)} \text{ with } n = 0, \pm 1, \pm 2, \ldots$$

For Y_β: Similarly, $\boxed{Y_m = C_m \sin\left(\frac{m\pi}{b}y\right)}$ and $\boxed{\beta = \frac{im\pi}{b}}$

with $m = 0, \pm 1, \pm 2, \ldots$

For Z_γ: $Z_\gamma(z=c) = 0 \Rightarrow \underbrace{E_\gamma e^{\gamma c}}_{G/2} + \underbrace{F_\gamma e^{-\gamma c}}_{-G/2} = 0$

$$\Rightarrow \begin{cases} E_\gamma = e^{-\gamma/2}\frac{G_\gamma}{2} \\ F_\gamma = -e^{+\gamma/2}\frac{G_\gamma}{2} \end{cases}$$ without loss of generality

$$\Rightarrow Z_\gamma(z) = G_\gamma\left[\frac{e^{-\gamma c}}{2}e^{\gamma z} - \frac{e^{\gamma c}}{2}e^{-\gamma z}\right]$$

$$= G_\gamma\left[\frac{e^{\gamma(z-c)} - e^{-\gamma(z-c)}}{2}\right]$$

$$= G_\gamma \sinh[\gamma(z-c)]$$

$$\Rightarrow Z_\gamma(z) = G_\gamma \sinh[\gamma(z-c)] \quad \text{with } \gamma^2 = -(\alpha^2+\beta^2)$$

$$= \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

$$\boxed{Z_{mn}(z) = G_{mn}\sinh[\gamma_{mn}(z-c)]}$$

$$\Rightarrow \boxed{\gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}$$

Thus

$$V(x,y,z) = \sum_{m,n=1}^{\infty} \underbrace{G_{mn}A_n C_m}_{\text{rename } V_{mn}} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh[\gamma_{mn}(z-c)]$$

Final boundary condition : $V(x, y, z=0) = \underbrace{V_0(x,y)}_{\text{not yet specified / given}}$

$$\Rightarrow V_0(x,y) = \sum_{m,n=1}^{\infty} V_{mn} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh\left[\gamma_{mn}(\cancel{z}^{\,0} - c)\right]$$

$$= -\sum_{m,n=1}^{\infty} V_{mn} \sinh(\gamma_{mn} c) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right)$$

$\Rightarrow$ This is a Fourier series representation of $V_0(x,y)$!!

Q: How do we determine V_{mn} ?

Fourier basis functions

A: $$\int_0^a \int_0^b V_0(x,y) \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{m'\pi}{b}y\right) dx\, dy$$

$$= -\sum_{m,n=1}^{\infty} V_{mn} \sinh(\gamma_{mn} c) \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{n'\pi x}{a}\right) dx \times \int_0^b \sin\left(\frac{m\pi}{b}y\right) \sin\left(\frac{m'\pi}{b}y\right) dy$$

substitution: $\begin{cases} u = \frac{\pi x}{a}, & du = \frac{\pi}{a} dx \\ v = \frac{\pi}{b} y, & dv = \frac{\pi}{b} dx \end{cases}$

$$= -\sum_{m,n=1}^{\infty} V_{mn} \sinh(\gamma_{mn} c) \; \underbrace{\frac{a}{\pi} \int_0^{\pi} \sin(nu) \sin(n'u)\, du}_{= \pi/2\, \delta_{nn'} \text{ orthogonality relation}} \times \underbrace{\frac{b}{\pi} \int_0^{\pi} \sin(mv) \sin(m'v)\, dv}_{= \pi/2\, \delta_{mm'} \text{ orthogonality relation}}$$

Important: $\int_0^{\pi} \sin(nx) \sin(mx)\, dx = \frac{\pi}{2} \delta_{mn}$ orthogonality relation

(use: $\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$\Rightarrow$ The sum dissappears !!

$$= -\frac{ab}{4} V_{m'n'} \sinh(\gamma_{m'n'} c)$$

drop the "primes"

$$\Rightarrow \boxed{V_{mn} = \frac{-4}{ab \sinh(\gamma_{mn} c)} \int_0^a \int_0^b V_0(x,y) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) dx dy}$$

with $\gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$

stopped here

Example: Special Case with $V_0(x,y) = V_0 = cst$

$$V_{mn} = \frac{-4 V_0}{ab \sinh(\gamma_{mn} c)} \int_0^a \sin\left(\frac{n\pi}{a} x\right) dx \int_0^b \sin\left(\frac{m\pi}{b} y\right) dy$$

note:

$$\int_0^a \sin\left(\frac{n\pi}{a} x\right) dx = -\left(\frac{a}{n\pi}\right) \cos\left(\frac{n\pi}{a} x\right)\Big|_0^a = -\frac{a}{n\pi} \underbrace{\left[\cos(n\pi) - 1\right]}_{= \begin{cases} -2 & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases}}$$

$$= \begin{cases} \frac{2a}{n\pi} & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases}$$

Thus
$$V_{mn} = \frac{-4 V_0}{\cancel{ab} \sinh(\gamma_{mn} c)} \left(\frac{2\cancel{a}}{n\pi}\right)\left(\frac{2\cancel{b}}{m\pi}\right) \quad \text{for } m,n = \text{odd}$$

$$= \frac{-16}{nm\pi^2} \frac{V_0}{\sinh(\gamma_{mn} c)}$$

Thus, finally, we get

$$\boxed{V(x,y,z) = \sum_{\substack{m,n=1\\(\text{odd})}}^{\infty} \frac{-16}{nm\pi^2} \frac{V_0}{\sinh(\gamma_{mn} c)} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \sinh\left(\gamma_{mn}(z-c)\right) \quad \text{with } \gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}$$

denominators grow for $n, m \to$ large

$\Longrightarrow$ Solution is an <u>infinite series</u>.

<u>Note</u>: You can construct solutions for more complicated arrangements based on the above solution: