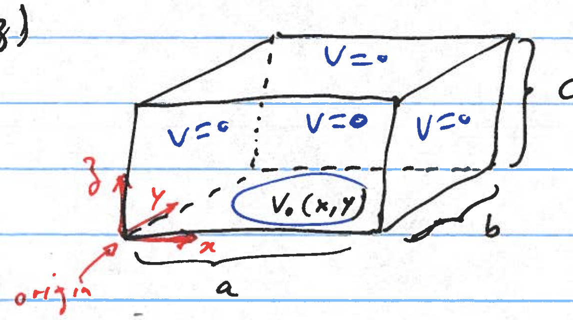
Monday, March 20, 2023

SEPARATION of Variables : Cartesian Symmetry (continued)

Example (continued) : Rectangular box with 5 sides at potential $V=0$ and one side at potential $V_0(x,y)$.

Q: What is $V(x,y,z)$ inside Box ?



Solution (see previous lecture) :

lots of work !!

$$\boxed{V(x,y,z) = \sum_{m,n=1}^{\infty} V_{mn} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh\left[\gamma_{mn}(z-c)\right]}$$

with $V_{mn} = \dfrac{-4}{ab \sinh(\gamma_{mn} c)} \displaystyle\int_0^a \int_0^b V_0(x,y) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) dx\,dy$

with $\gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$

Example: Special case with $V_0(x,y) = V_0 = \text{constant}$

$$\Rightarrow V_{mn} = \frac{-4V_0}{ab\sinh(\gamma_{mn}c)} \int_0^a \sin\left(\frac{n\pi}{a}x\right)dx \int_0^b \sin\left(\frac{m\pi}{b}y\right)dy$$

note: $\displaystyle\int_0^a \sin\left(\frac{n\pi}{a}x\right)dx = -\left(\frac{a}{n\pi}\right)\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a = -\frac{a}{n\pi}\underbrace{\left[\cos(n\pi) - 1\right]}_{= \begin{cases} -2 & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even}\end{cases}}$

$$\Rightarrow \int_0^a \sin\left(\frac{n\pi}{a}x\right)dx = \begin{cases} \dfrac{2a}{n\pi} & \text{for } n = \text{odd} \\[2ex] 0 & \text{for } n = \text{even} \end{cases}$$

Thus $V_{mn} = \dfrac{-4\,V_0}{\cancel{ab}\,\sinh(\gamma_{mn}c)}\left(\dfrac{2\cancel{a}}{n\pi}\right)\left(\dfrac{2\cancel{b}}{m\pi}\right)$ for $m,n = \text{odd}$

$$= \frac{-16}{nm\pi^2}\,\frac{V_0}{\sinh(\gamma_{mn}c)}$$

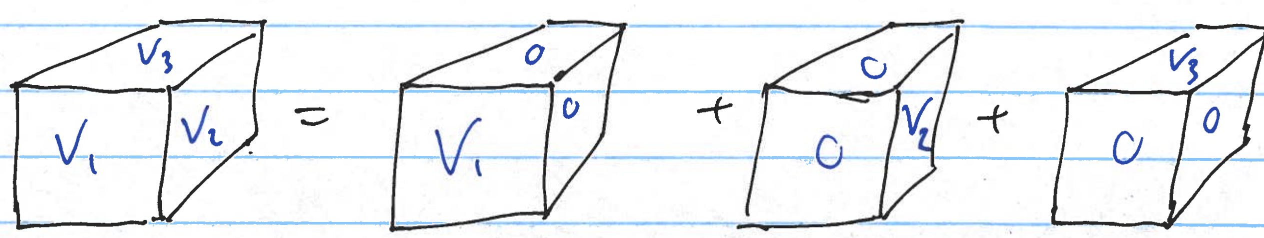
Thus finally we get

$$\boxed{V(x,y,z) = \sum_{\substack{m,n=1\\(\text{odd})}}^{\infty} \frac{-16\,V_0}{\underbrace{nm\pi^2}\,\underbrace{\sinh(\gamma_{mn}c)}} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\gamma_{mn}(z-c)\right) \quad \text{with } \gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}$$

denominators grow for $n, m \to$ large

$\Rightarrow$ Solution is an <u>infinite series</u>.

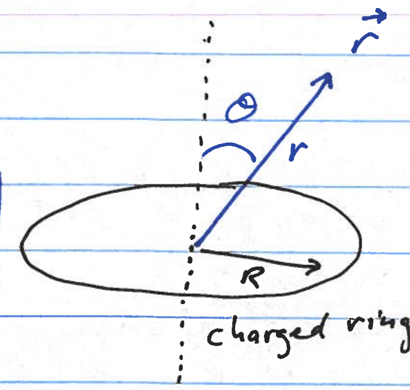
<u>Note</u>: You can construct solutions for more complicated arrangements based on the above solution:



Separation of variables : Spherical Symmetry

Simplest case : Azimuthal Symmetry

We expect $V(r, \theta, \phi) = V(r, \theta)$



In spherical coordinates: $\boxed{\nabla^2 V(r, \theta) = 0}$

Laplace's eq.

$$\Longleftrightarrow \quad \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

sometimes useful: $\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)$

If we assume $V(r,\theta) = R(r)\,\Theta(\theta)$ as the separable solution, then we get

[multiply by r^2]

$$\Longrightarrow \quad \Theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + R \frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\Theta\right) = 0$$

R(r) $\quad$ $\Theta(\theta)$

divide by $R(r)\,\Theta(\theta)$

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)}_{\text{Constant} = C_r} + \underbrace{\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\Theta\right)}_{\text{Constant} = C_\theta} = 0$$

[Constant since each term is independent of the other]

<u>note</u>: $C_r + C_\theta = 0 \iff C_r = -C_\theta = k(k+1)$

by convention

for $R(r)$: $\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R(r)\right) = k(k+1)\, R(r)$

↳ solution: $\boxed{R_k(r) = A_k r^k + B_k r^{-(k+1)}}$

for $H(\theta)$: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial H}{\partial\theta}\right) = -k(k+1)\, H$

↳ convenient change of variable: $x = \cos\theta$

NOT coordinate x

↳ then $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial x}\underbrace{\frac{\partial x}{\partial\theta}}_{-\sin\theta} = -\sin\theta\frac{\partial}{\partial x}$

Thus the differential equation becomes:

$$\underbrace{\frac{1}{\cancel{\sin\theta}}(-\cancel{\sin\theta})}_{-1}\frac{\partial}{\partial x}\left(\underbrace{\sin\theta\,(-\sin\theta)}_{-\sin^2\theta}\frac{\partial}{\partial x} H(x)\right) = -k(k+1)\, H(x)$$

$$-\sin^2\theta = \cos^2\theta - 1 = x^2 - 1$$

$$\Leftrightarrow \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x} H(x)\right] = -k(k+1)\, H(x)$$

$$\Leftrightarrow \boxed{(1-x^2)\frac{\partial^2}{\partial x^2} H(x) - 2x\frac{\partial}{\partial x} H(x) + k(k+1)\, H(x) = 0}$$

Legendre differential equation

Solution: $H(x) = C_k P_k(x) + D_k Q_k(x)$

$$\Rightarrow \Theta(\theta) = C_k \underbrace{P_k(\cos\theta)} + D_k \underbrace{Q_k(\cos\theta)}$$

Legendre function of the first kind

↳ does not diverge over $0 \le \theta \le \pi$

for $k = 0, 1, 2, 3, \ldots$

↳ $k \to \ell$ = positive integers

2nd kind

$$\lim_{\substack{\theta \to 0 \\ \text{or} \\ \theta \to \pi}} Q_k(\cos\theta) \to \infty$$

↳ the divergence generally eliminates this type of solution.

note: negative ℓ integers are "ok", but $P_{-\ell-1}(x) = P_\ell(x)$

Thus, generally, we only consider solutions in terms of the Legendre polynomials: $\Theta(\theta) = P_\ell(\cos\theta)$

with $\ell = 0, 1, 2, 3, \ldots$

Legendre Polynomials

Rodrigues formula: $P_\ell(x) = \dfrac{1}{2^\ell \ell!}\left(\dfrac{d}{dx}\right)^\ell (x^2-1)^\ell$

$P_0(x) = 1$ even

$P_1(x) = x$ odd

$P_2(x) = (3x^2-1)/2$ even

$P_3(x) = (5x^3-3x)/2$ odd

$P_4(x) = (35x^4 - 30x + 3)/8$ even

$P_5(x) = (63x^5 - 70x^3 + 15x)/8$ odd

<u>General Solution</u>: $$V(r,\theta) = \sum_{\ell=0}^{\infty} \overbrace{\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right)}^{R(r)} \overbrace{P_\ell(\cos\theta)}^{\Theta(\theta)}$$

$A_\ell r^\ell$: diverges for $r \to +\infty$ ↳ use near $r \to 0$

$B_\ell r^{-(\ell+1)}$: diverges for $r \to 0$ ↳ use for $r \to +\infty$

In order to avoid divergences at $r \to 0$ & $r \to +\infty$, we consider the solution:

$$V(r,\theta) = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} C_\ell \left(\frac{r}{R}\right)^\ell P_\ell(\cos\theta) & \text{for } r \leq R \\ \displaystyle\sum_{\ell=0}^{\infty} C_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta) & \text{for } r \geq R \end{cases}$$

same coefficients ensures continuity at $r = R$

$\underline{\underline{R}}$ can be any finite radius, but it is generally suggested by the particular problem.

⚠ If there is a boundary at $r = R$, then there may be a boundary condition to satisfy.

<u>Example</u>: Spherical shell (non-conducting) with charge density $\sigma(\theta) = \sigma_0 \cos\theta$ and radius R.

→ Calculate $V(r,\theta)$ everywhere $\left(\begin{matrix}\text{inside} \\ \& \\ \text{outside}\end{matrix}\right)$.