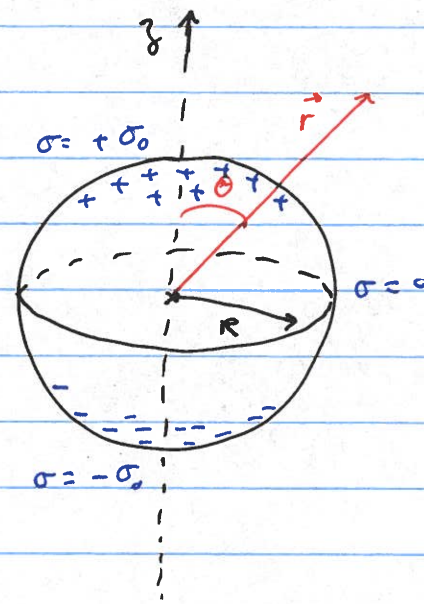
Wednesday, March 22, 2023

Separation of variables : Spherical coordinates / symmetry (continued)

Example: Spherical shell (non-conducting) with charge density $\sigma(\theta) = \sigma_0 \cos\theta$ and radius R.

→ Calculate $V(r, \theta)$ everywhere.
(inside & outside)



direct method: $V(r,\theta) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int \frac{\sigma(\theta')}{|\vec{r} - \vec{r}\,'|} \underbrace{r'^2}_{R^2} \sin\theta' \underbrace{dr'}_{\text{no } dr' \text{ integral}} d\theta' d\phi'$

↳ somewhat messy

Separation of variables:

inside shell $(r < R)$: $V_{in}(r,\theta) = \displaystyle\sum_{\ell=0}^{\infty} C_\ell \left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta)$

outside shell $(r > R)$: $V_{out}(r,\theta) = \displaystyle\sum_{\ell=0}^{\infty} C_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta)$

Boundary condition : At shell surface $\begin{cases} E_\perp \text{ is discontinuous} \\ E_{\parallel} \text{ is continuous} \end{cases}$

[see Lecture 8, Feb. 20]

$$\Delta E_\perp = \frac{\sigma_s}{\varepsilon_0} \Rightarrow E_{out,r}\Big|_{r=R} - E_{in,r}\Big|_{r=R} = \frac{\sigma(\theta)}{\varepsilon_0} = \frac{\sigma_0 \cos\theta}{\varepsilon_0}$$

$E_r = -\frac{\partial V}{\partial r}$

$$\Leftrightarrow \frac{\partial V_{out}}{\partial r}\Big|_{r=R} - \frac{\partial V_{in}}{\partial r}\Big|_{r=R} = -\frac{\sigma_0 \cos\theta}{\varepsilon_0}$$

$$\Rightarrow \sum_{l=0}^{\infty} C_l \frac{R^{l+1}}{r^{l+2}}(-l-1)P_l(\cos\theta)\Big|_{r=R} - \sum_{l=0}^{\infty} \frac{C_l\, l}{R^l} r^{l-1} P_l(\cos\theta)\Big|_{r=R} = -\frac{\sigma_0 \cos\theta}{\varepsilon_0}$$

$r=R$

$$\Rightarrow \sum_{l=0}^{\infty} \frac{C_l(-l-1)}{R} P_l(\cos\theta) - \sum_{l=0}^{\infty} \frac{C_l\, l}{R} P_l(\cos\theta) = -\frac{\sigma_0 \cos\theta}{\varepsilon_0}$$

$$\Leftrightarrow \sum_{l=0}^{\infty} \frac{C_l(2l+1)}{R} P_l(\cos\theta) = \frac{\sigma_0 \cos\theta}{\varepsilon_0}$$

Q: How do we determine C_l's?

A: Apply Fourier's trick! (for Legendre Polynomials)

$\begin{cases} \text{multiply both sides by } P_{l'}(\cos\theta) \\ \text{integrate both sides by } dx = d(\cos\theta) \end{cases}$

$$\Rightarrow \int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta)\left[\sum_{\ell=0}^{\infty} \frac{C_\ell}{R}(2\ell+1) P_\ell(\cos\theta)\right] d(\cos\theta)$$

$$= \int_{\cos\theta=-1}^{\cos\theta=+1} \frac{\sigma_0 \cos\theta}{\varepsilon_0} P_{\ell'}(\cos\theta)\, d(\cos\theta)$$

$$\Leftrightarrow \sum_{\ell=0}^{\infty} \frac{C_\ell}{R}(2\ell+1) \underbrace{\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta)}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}} .$$

$$= \frac{\sigma_0}{\varepsilon_0} \int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) \cos\theta\, d(\cos\theta)$$

<u>formula</u>:

$$\boxed{\begin{gathered}\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'} \\ \text{or} \quad \int_{x=-1}^{x=+1} P_{\ell'}(x) P_\ell(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}\end{gathered}}$$

$\Rightarrow$ sum dissappears!

$$\Rightarrow \frac{C_{\ell'}}{R}(2\ell'+1)\frac{2}{2\ell'+1} = \frac{\sigma_0}{\varepsilon_0} \underbrace{\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta) \overbrace{\cos\theta}^{P_1(\cos\theta)}\, d(\cos\theta)}_{\frac{2}{2\ell'+1}\delta_{\ell'1} = \frac{2}{2(1)+1} = \frac{2}{3}}$$

$\Rightarrow \ell' = 1$

$\Rightarrow$ for $\ell' \neq 1$, $C_{\ell'} = 0$

<u>Note</u>: The entire sum dissappears! (this does not always happen)

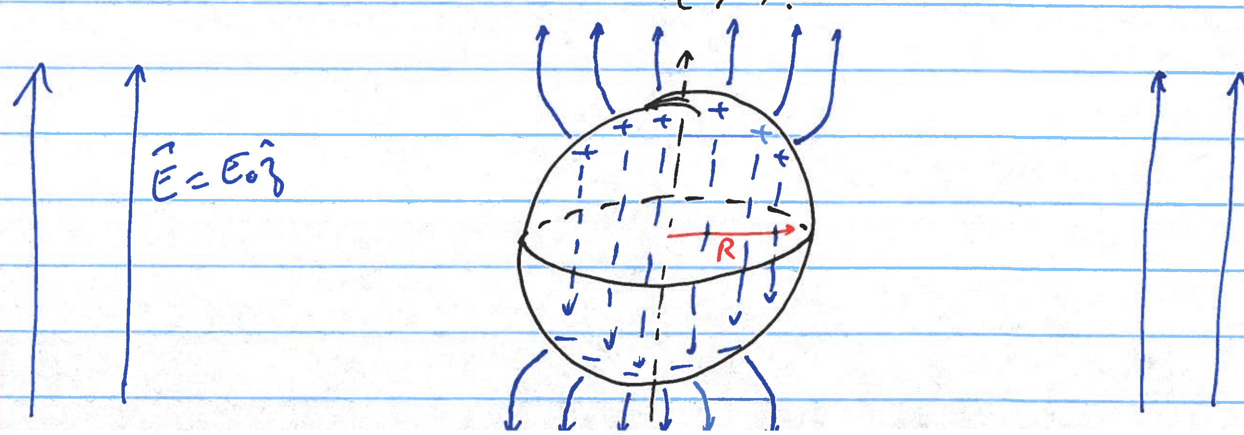
for $\ell' = 1$, $\quad \frac{C_1}{R}\cancel{2} = \frac{\sigma_0}{\varepsilon_0}\frac{\cancel{2}}{3}$

$$\Rightarrow \boxed{C_1 = \frac{\sigma_0}{\varepsilon_0}\frac{R}{3}}$$

$$\Rightarrow \begin{cases} V_{in} = C_1\left(\frac{r}{R}\right)^1 P_1(\cos\theta) = \frac{\sigma_0}{\varepsilon_0}\frac{\cancel{R}}{3}\frac{r}{\cancel{R}}\cos\theta \\ V_{out} = C_1\left(\frac{R}{r}\right)^{1+1} P_1(\cos\theta) = \frac{\sigma_0}{\varepsilon_0}\frac{R}{3}\frac{R^2}{r^2}\cos\theta \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} V_{in}(r,\theta) &= \frac{\sigma_0}{3\varepsilon_0} r\cos\theta = \frac{\sigma_0}{3\varepsilon_0} z && (r \le R) \\ V_{out}(r,\theta) &= \frac{\sigma_0 R^3}{3\varepsilon_0}\frac{\cos\theta}{r^2} && (r \ge R) \end{aligned}}$$

Example 2: Solid metal sphere (conducting) in an otherwise uniform electric field $\vec{E} = E_0\hat{z}$.
↳ Calculate $V(r,\theta)$.



Q1: what is the potential at $|\vec{r}| \to +\infty$?
A1: Not zero, Actually $V(r\to\infty) \to \pm\infty, 0$!!!
↳ unusual.

Q2: what is the potential inside the metal sphere ?
A2: It must be an equipotential!
↳ $V_{\substack{\text{inside}\\ \text{sphere}}}(r,\theta) = cst$
$r<R$

For convenience, we pick $V_{sphere} = 0$ and put it at origin
inside + surface

Q3: what is the potential far from the sphere?
A3: $\vec{E}_{far} = E_0 \hat{z} \implies V = -E_0 z + cst$
$[\vec{E} = -\vec{\nabla} V]$
$cst = 0$ by symmetry for $V_{sphere} = 0$

$\implies$ Boundary Conditions:
(i) $V=0$ for $r \leq R$
(ii) $V \to -E_0 z$ for $r \gg R$
$= -E_0 r\cos\theta$
(iii) $\vec{E}(r=R)$ $\hat{r}$

General form of solution (outside, for $r \geq R$)

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left[a_\ell \left(\frac{r}{R}\right)^{\ell} + b_\ell \left(\frac{R}{r}\right)^{\ell+1} \right] P_\ell(\cos\theta)$$

boundary condition (i):
if $V(r=R,\theta)=0$, then $a_\ell \cancel{\left(\frac{R}{R}\right)^{\ell}}^{\,1} + b_\ell \cancel{\left(\frac{R}{R}\right)^{\ell+1}}^{\,1} = 0$

$$\implies \boxed{a_\ell = -b_\ell}$$

Thus $\underset{\text{outside}}{V(r,\theta)} = \sum_{\ell=0}^{\infty} a_\ell \left[\left(\frac{r}{R}\right)^{\ell} - \left(\frac{R}{r}\right)^{\ell+1} \right] P_\ell(\cos\theta)$

<u>boundary condition</u> (ii)

for $r \gg R$, then $V(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell \left[\left(\frac{r}{R}\right)^{\ell} - \underbrace{\left(\frac{R}{r}\right)^{\ell+1}}_{} \right] P_\ell(\cos\theta)$

$= -E_0 r \cos\theta$

individual terms do not contribute at $r \gg R$

Apply "Fourier trick": $\displaystyle\int_{\cos\theta=-1}^{\cos\theta=+1} P_{\ell'}(\cos\theta)\, P_\ell(\cos\theta)\, d(\cos\theta) = \frac{2}{2\ell+1}\,\delta_{\ell'\ell}$

<u>OR</u> just note that $\cos\theta = P_{\ell=1}(\cos\theta)$

$\hookrightarrow$ only $\ell=1$ term is present in sum: $\boxed{a_{\ell\neq 1} = 0}$

$\Rightarrow \; a_{\ell=1} \left(\frac{r}{R}\right)^{\ell=1} P_{\ell=1}(\cos\theta) = -E_0 r \cos\theta$

$\Leftrightarrow \; a_1 \frac{\cancel{r}}{R} \cancel{\cos\theta} = -E_0 \cancel{r} \cancel{\cos\theta} \;\Leftrightarrow\; \boxed{a_1 = -E_0 R}$

Thus

$$\boxed{\begin{aligned} V_{\text{outside}}(r,\theta) &= -E_0 R \left[\left(\frac{r}{R}\right) - \left(\frac{R}{r}\right)^2 \right] \cos\theta \\ &= -E_0 r \cos\theta + E_0 \frac{R^3}{r^2} \cos\theta \\ &= V_{ext} + E_0 \frac{R^3}{r^2} \cos\theta \end{aligned}}$$

Q: What's the surface $\underbrace{\text{charge}}_{\text{induced charge}}$ distribution on the sphere?

A: $$-\frac{\sigma(\theta)}{\varepsilon_0} = \left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} - \underbrace{\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R}}_{=0} = \Delta E$$

$$= -E_0 \cos\theta + E_0 R^3 \cos\theta \, (-2) \left.\frac{1}{r^3}\right|_{r=R}$$

$$= -E_0 \cos\theta \, (1+2)$$

$$= -3 E_0 \cos\theta \implies \boxed{\sigma(\theta) = 3 \varepsilon_0 E_0 \cos\theta}$$

$$\begin{cases} \sigma(\theta < \pi/2) & \text{is positive} \\ \sigma(\theta > \pi/2) & \text{is negative} \\ \sigma(\theta = \pi/2) & = 0 \end{cases}$$