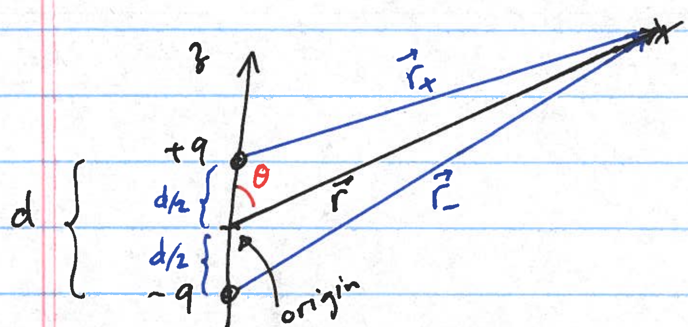
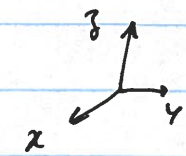
Monday, March 27, 2023

Multipole expansion & Dipole moments [chpt 3.4]

Objective: Describe the electric field at large distances.

Example: consider a standard dipole





$$\vec r_\pm = \vec r \mp \frac{d}{2}\hat z \Rightarrow \vec r_\pm^{\,2} = r^2 + \left(\frac{d}{2}\right)^2 \overbrace{\mp\, dr\cos\theta}^{2\vec r\cdot\left(\mp\frac{d}{2}\hat z\right)}$$

$$\Rightarrow r_\pm = r\sqrt{1+\left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta}$$

$$\Rightarrow \frac{1}{r_\pm} = \frac{1}{r}\,\frac{1}{\sqrt{1+\left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta}}$$

recall: $\sqrt{1+\varepsilon} = 1+\frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \dots$

limit $d \ll r$

$$\simeq \frac{1}{r}\,\frac{1}{1+\frac{1}{2}\left[\underbrace{\left(\frac{d}{2r}\right)^2}_{\text{very small} \,\to\, \text{neglect}} \mp \frac{d}{r}\cos\theta\right] - \underbrace{\frac{1}{8}\left[\left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta\right]^2}_{\text{very small} \,\to\, \text{neglect}} + \dots}$$

$$\simeq \frac{1}{r}\,\frac{1}{1 \mp \frac{d}{2r}\cos\theta} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$\frac{1}{1+\varepsilon} \simeq 1-\varepsilon$

$$\Leftrightarrow \quad \frac{1}{r_\pm} \simeq \frac{1}{r} \pm \frac{d}{2r^2}\cos\theta$$

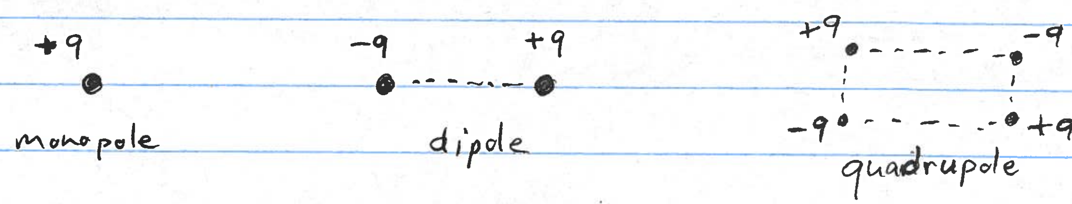
The potential is then

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} + \frac{-q}{r_-}\right) = \frac{q}{4\pi\varepsilon_0}\left\{\cancel{\frac{1}{r}} + \frac{d}{2r^2}\cos\theta - \left[\cancel{\frac{1}{r}} - \frac{d}{2r^2}\cos\theta\right]\right\}$$

$$\Rightarrow \quad \boxed{V(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}}$$

(to lowest order in $\frac{d}{r}$)

<u>Long distance Behavior</u> (<u>electrostatics</u>):



monopole	dipole	quadrupole
$V \sim \frac{1}{r}$	$V \sim \frac{1}{r^2}$	$V \sim \frac{1}{r^3}$
$\lvert\vec{E}\rvert \sim \frac{1}{r^2}$	$\lvert\vec{E}\rvert \sim \frac{1}{r^3}$	$\lvert\vec{E}\rvert \sim \frac{1}{r^4}$

<u>Note</u>: – <u>Bare charges</u> are not that common in nature, since they attract opposing charges to cancel out.

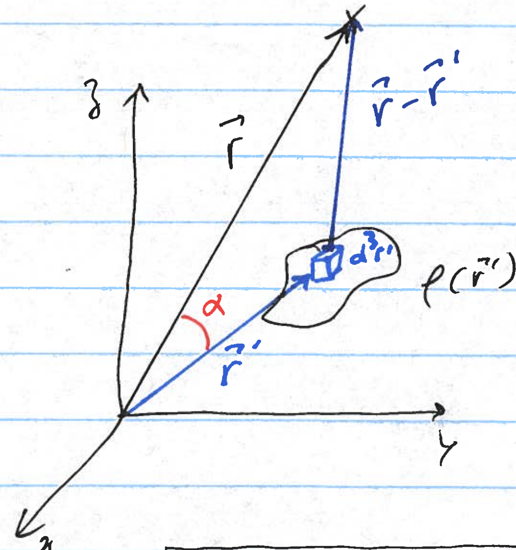
– <u>Electric dipoles</u> are very common (typically electrically neutral)

Multipole Expansion

Consider a charge distribution $\rho(\vec{r})$.

Q: what is its electric field at large distances (far away)?

A: Use the multipole expansion!



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

law of cosines: $|\vec{r}-\vec{r}\,'| = \sqrt{(\vec{r}-\vec{r}\,')^2} = \sqrt{\vec{r}^{\,2} + \vec{r}\,'^{2} - 2\vec{r}\cdot\vec{r}\,'}$

$$= r^2 + r'^2 - 2rr'\cos\alpha$$

$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\alpha}$$

Thus $$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\,\underbrace{\frac{1}{\left[1+\left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\alpha\right]^{1/2}}}$$

$$\frac{1}{\sqrt{1-\varepsilon}} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots$$

$$\left(\text{for } \varepsilon = \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\alpha \ll 1\right)$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\Big\{1 - \frac{1}{2}\left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right] + \frac{3}{8}\left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]^2$$

$$- \frac{5}{16}\left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]^3 + \cdots \Big)$$

$$= \frac{1}{r}\Big\{ \underbrace{1}_{P_0(\cos\alpha)} + \left(\frac{r'}{r}\right)\underbrace{\cos\alpha}_{P_1(\cos\alpha)} + \left(\frac{r'}{r}\right)^2 \underbrace{\frac{3\cos^2\alpha - 1}{2}}_{P_2(\cos\alpha)}$$

$$+ \left(\frac{r'}{r}\right)^3 \underbrace{\frac{5\cos^3\alpha - 3\cos\alpha}{2}}_{P_3(\cos\alpha)} + \cdots \Big\}$$

infer, intuit, guess

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$\Rightarrow \boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)}$$

for $r' \ll r$
($r' < r$ is sufficient)

$\cos\alpha = \hat{r}\cdot\hat{r}\,'$

Thus

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r' \\ &= \frac{1}{4\pi\epsilon_0}\Big\{ \frac{1}{r}\underbrace{\int \rho(\vec{r}\,')\, d^3r'}_{\text{monopole term}} + \frac{1}{r^2}\underbrace{\int r'\rho(\vec{r}\,')\cos\alpha\, d^3r'}_{\text{dipole term}} \\ &\quad + \underbrace{\frac{1}{r^3}\int r'^2 \rho(\vec{r}\,')\left[\frac{3\cos^2\alpha - 1}{2}\right] d^3r'}_{\text{quadrupole term}} + \cdots \Big\} \end{aligned}}$$

Multipole expansion of potential
(for $r > r'$)

<u>note</u>: $\int \rho(\vec{r}\,')\, d^3r' = Q$ = total charge

The electric dipole

If the total charge is zero (i.e. $Q=0$), then the to lowest order the field/potential is dipole like:

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\rho(\vec{r}')\underbrace{\cos\alpha}_{\hat{r}\cdot\hat{r}'}\, d^3r'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\cdot\underbrace{\int \vec{r}'\rho(\vec{r}')\,d^3r'}_{\text{electric dipole moment}}$$

definition:

$$\boxed{\text{electric dipole moment} = \vec{p} = \int_{\substack{\text{all space} \\ \text{or} \\ \text{volume of } \rho(\vec{r}')}} d^3r'\,\rho(\vec{r}')\,\vec{r}'}$$

$$\Rightarrow \boxed{V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$$

Q: Is $\vec{p}$ an intrinsic quantity independent of the coordinate system?

consider the electric dipole moment $\vec{p} = \int d^3r'\,\rho(\vec{r}')\vec{r}'$

↳ now shift $\rho(\vec{r})$ by $\vec{d}$, i.e. $\rho_d(\vec{r}) = \rho(\vec{r}-\vec{d})$

$$\Rightarrow \vec{p}_d = \int d^3r'\,\rho(\vec{r}'-\vec{d})\,\vec{r}' \underset{\substack{\text{substitution } \vec{r}''=\vec{r}'-\vec{d} \\ d^3r''=d^3r'}}{=} \int d^3r''\,\rho(\vec{r}'')(\vec{r}''+\vec{d})$$

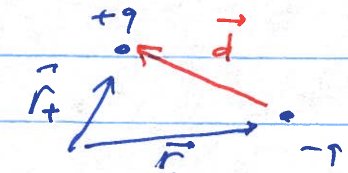
$$= \overbrace{\int d^3r''\,\rho(\vec{r}'')\,\vec{r}''}^{\vec{p}} + \vec{d}\underbrace{\int d^3r''\rho(\vec{r}'')}_{Q}$$

Thus $\vec{P}_d = \vec{P} + Q\vec{d}$

A: $\Rightarrow$ The dipole moment depends on the ~~origin~~ choice of origin!

$\Rightarrow$ For a neutral system, the dipole moment is <u>independent</u> of origin (i.e. it's coordinate free).

<u>Ex:</u> 2 equal & opposite point charges



$$\vec{p} = q\vec{r}_+ - q\vec{r}_-$$
$$= q(\vec{r}_+ - \vec{r}_-)$$
$$= q\vec{d} \Rightarrow \boxed{\vec{p} = q\vec{d}}$$

$\vec{d}$ points from "$\underline{\underline{-q}}$" to "$\underline{\underline{+q}}$"

<u>Ex:</u> N point charges

$$\boxed{\vec{p} = \sum_{i=1}^{N} q_i \vec{r}_i}$$

e.g. atom: nucleus + e^-

molecule: H_2O in a weak E-field

<u>Electric Field of a dipole</u>: $\vec{E} = -\vec{\nabla} V = -\vec{\nabla}\left(\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}\right)$

some algebra

Coordinate free form
↳ useful

$$\Rightarrow \boxed{\vec{E}_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}}{r^3}}$$

some algebra

$$\Rightarrow \vec{E}_{dipole}(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\left[\frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right]$$

{for $\vec{p} = p\hat{z}$}