Physics 430 and 631: Quantum Optics & Atomics
Due date: Thursday, September 4, 2025

Problem Set #1

I. Coherence 101: Hanbury-Brown and Twiss Experiment

Read the Hanbury-Brown and Twiss paper and the associated letter by E. Purcell.
1. Write a paragraph explaining the Brown-Twiss effect using classical physics.
2. Write a paragraph explaining the Brown-Twiss effect using quantum physics.

II. Semi-classical atom

Consider a semi-classical atom consisting of a very heavy nucleus with charge $+q_e$ connected by a spring to an electron with charge $-q_e$ mass m_e. This atom behaves as a harmonic oscillator with natural frequency ω_0 (value undefined at this point, but assumed to be very high frequency, e.g., 10^{14} Hz).

This harmonic oscillator experiences radiation damping, such that when excited the radial position r of the electron decays as $r(t) = r_0 e^{-\gamma t} \sin(\omega_0 t)$, where r_0 is the equilibrium position of the electron, and γ is the damping constant.

Electromagnetically driven atom

Next, we direct an electromagnetic plane wave at this atom (located at the origin). The electric field of the plane wave is given by

$$\vec{E}(t) = E_0 \cos(ky - \omega t)\,\hat{z}\ ,$$

where E_0 is the amplitude of the wave, ω is the frequency of the wave (in rads/s), $k = \omega/c$ is the wavenumber (i.e., "wavevector"), and c is the speed of light in vacuum. We will ignore any magnetic forces associated with this wave.

1. Show that the equation of motion for the atom's electron's position $z(t)$ is given by:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = -\frac{q_e}{m_e} E_0 \cos(\omega t)$$

2. Ignoring any transient behavior, show that the steady state solution is oscillatory with the following form

$$z(t) = Z_0 \cos(\omega t - \phi)\ ,$$

where the amplitude is given by

$$Z_0 = \frac{-(q_e/m_e)E_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2}}$$

and the phase is given by

$$\phi = \tan^{-1}\left(\frac{2\omega\gamma}{\omega_0^2 - \omega^2}\right)$$

3. Sketch plots of $Z_0(\omega)$ and $\phi(\omega)$ in the vicinity of $\omega = \omega_0$ over a range of roughly $\pm 5\gamma$.

4. This driven atom will emit dipole radiation. The total average optical power P radiated by an oscillating dipole is the following:

$$P = \frac{1}{4\pi\varepsilon_0}\frac{q_e^2\omega^4}{3c^3}Z_0^2$$

where ε_0 is the permittivity of free space.

We now consider the case of a driven optical transition with $\omega_0 \approx 2\pi \times 10^{14}$ Hz and $\gamma \approx 2\pi \times 10$ MHz. Show that for this parameter range, the radiated power can be written as

$$P \approx \frac{E_0^2}{16\pi\varepsilon_0}\frac{q_e^4}{3m_e^2c^3}\cdot\frac{\omega_0^2}{\delta^2+\gamma^2}$$

where $\delta = \omega - \omega_0$ is the detuning of the drive frequency from the transition.

5. The last fraction in the expression for the radiated power P in (4) is a Lorentzian function.

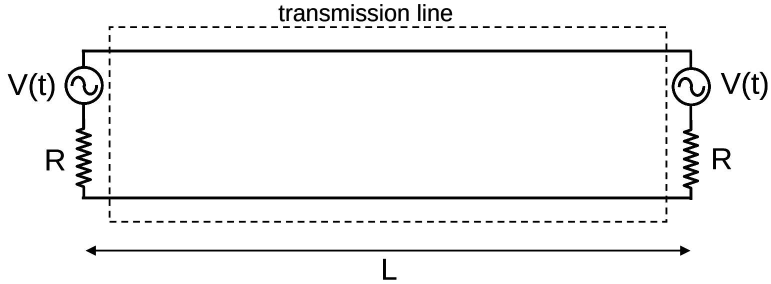
- Sketch a plot of this Lorentzian versus δ (e.g., over a range of roughly $\pm 5\gamma$).
- Calculate the "full width at half max" (FWHM) for this curve.

---------------- *Graduate student problem* ---------------

III. Classical noise: Nyquist's derivation of Johnson Noise
(example of the Fluctuation-Dissipation Theorem)

Thermal agitation of electrons produces the resistance to electrical conduction in a resistor with resistance R. This same thermal agitation moves the electrons around randomly inside the resistor so as to produce a small fluctuating voltage, V(t), across the resistor terminals, which is referred to as Johnson noise. Since R and V(t) are produced by the same phenomena (thermal agitation of electrons), they are also related. In this problem, you will derive the relation between V(t), R, and T (temperature).

We will model a resistor with a fluctuating voltage, V(t), across its terminals as an ideal resistor in series with a fluctuating signal generator. In Nyquist's derivation, two identical "noisy" resistors with resistance R are connected via a transmission line with impedance R, so as to produce a 1-d electrical circuit equivalent of the blackbody radiation problem, as shown in the figure below. The impedance R of the transmission line guarantees that signals generated on one end of the transmission line will be completely absorbed at the other end, without reflections.



1. Transmission line modes

The boundary condition for the transmission line is that the electromagnetic field must have a node at either end of the transmission line. The speed of light in the transmission line is c_t.

a. Calculate the permitted wavelengths of the modes of the electromagnetic field of the transmission line.

b. Calculate the permitted frequencies of the modes of the electromagnetic field of the transmission line.

c. Show that over a large frequency span, Δf, the number of modes of the electromagnetic field is $N(\Delta f) = (2L/c_t)\Delta f$.

2. Thermal population of the electromagnetic modes

According to the equipartition theorem, each mode of the electromagnetic field (i.e. degree of freedom) has a total energy of kT stored in it, where k is Boltzmann's constant. This energy comes from the two resistors which are both at temperature T.

a. How long does it take for thermal energy emitted by one resistor to arrive at the other resistor (and be absorbed)?

b. Show that the electromagnetic power $dP(f)$, in a frequency band df, absorbed by a resistor is $dP(f)=kTdf$.

3. Johnson noise

In thermal equilibrium, the power absorbed by a resistor in a given frequency range is also the power emitted by the resistor in the same frequency range due to the fluctuating voltage, $V(t)$, on its terminals.

a. Calculate the current I generated by the fluctuating voltage source on one of the two resistors.

b. Calculate the electrical power dissipated in one resistor due to the current generated by the fluctuating voltage source of the other resistor.

c. Show that over a frequency range Δf, the RMS value of the fluctuating voltage (i.e. Johnson noise) from a single resistor must be given by the expression:

$$\langle V_{RMS} \rangle = \sqrt{4RkT\Delta f}$$

d. Calculate the RMS Johnson voltage noise for a 10 MΩ resistor at room temperature over a 1 kHz bandwidth.