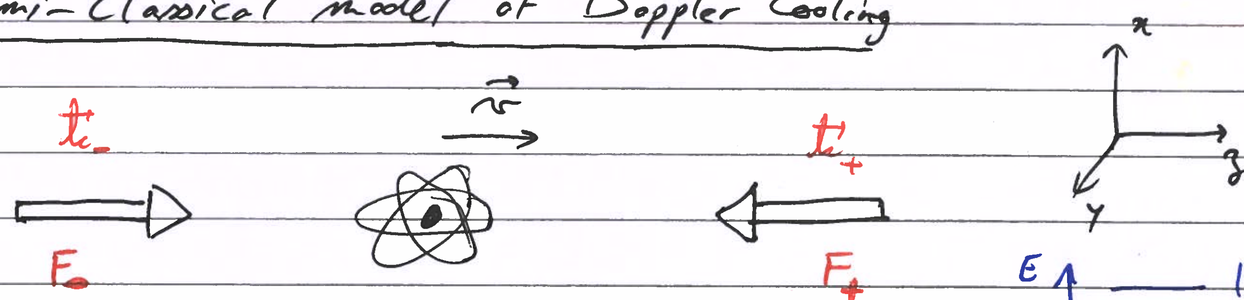
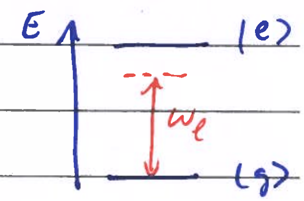
Thursday, November 13, 2025

Semi-Classical model of Doppler Cooling





$$\vec{F} = \frac{d}{dt}\vec{p} = \text{momentum} \times \frac{\text{kicks}}{\text{second}}$$

$$= \hbar k \, \gamma_{scattering}$$

$s_0 = \frac{I}{I_{sat}}$

$$= \hbar k \frac{s_0}{1 + s_0 + \left(\frac{2\delta}{\gamma}\right)^2} \frac{\gamma}{2}$$

$2\pi \times 6$ MHz for Rb

$I_{sat} = 1.6 \text{ mW}/\text{cm}^2$ for Rb

1D model: We will do the calculation in the low intensity limit ($s_0 = \frac{I}{I_{sat}} \ll 1$) so that the atoms do not have to choose between the two laser beams, i.e. saturation is not important.

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detuning: $\quad \delta = \delta_\ell \pm \underbrace{\Delta\omega_{Doppler}}_{\vec{k}\cdot\vec{v}} = \delta_\ell \pm kv$

laser direction

$\delta_\ell = \omega_\ell - \omega_{eg}$

In 1D: $\quad F_{total} = -F_+ + F_-$

$$= \hbar k \frac{\gamma}{2} s_0 \left\{ \frac{-1}{1 + s_0 + \left[\frac{2(\delta_\ell + kv)}{\gamma}\right]^2} + \frac{1}{1 + s_0 + \left[\frac{2(\delta_\ell - kv)}{\gamma}\right]^2} \right\}$$

Low v damping force:

Assume $|kv| \ll |\delta|$

$$F = \frac{\hbar k \gamma s_0}{2}\left\{ \frac{-1}{1+s_0+\frac{4\delta_\ell^2}{\gamma^2}+\frac{8kv\delta_\ell}{\gamma^2}+\underbrace{\cancel{4\left(\frac{kv}{\gamma}\right)^2}}_{\text{neglect}}} + \frac{1}{1+s_0+\frac{4\delta_\ell^2}{\gamma^2}-\frac{8kv\delta_\ell}{\gamma^2}+\underbrace{\cancel{4\left(\frac{kv}{\gamma}\right)^2}}_{\text{neglect}}} \right\}$$

$$\simeq \frac{\hbar k \gamma s_0}{2}\left\{ \frac{-1}{\left(1+s_0+\frac{4\delta_\ell^2}{\gamma^2}\right)\underbrace{\left(1+\frac{8kv\delta_\ell/\gamma^2}{1+s_0+4\delta_\ell^2/\gamma^2}\right)}_{\frac{1}{1+\varepsilon}\simeq 1-\varepsilon}} + \frac{1}{\left(1+s_0+\frac{4\delta_\ell^2}{\gamma^2}\right)\underbrace{\left(1-\frac{8kv\delta_\ell/\gamma^2}{1+s_0+4\delta_\ell^2/\gamma^2}\right)}_{\frac{1}{1-\varepsilon}\simeq 1+\varepsilon}} \right\}$$

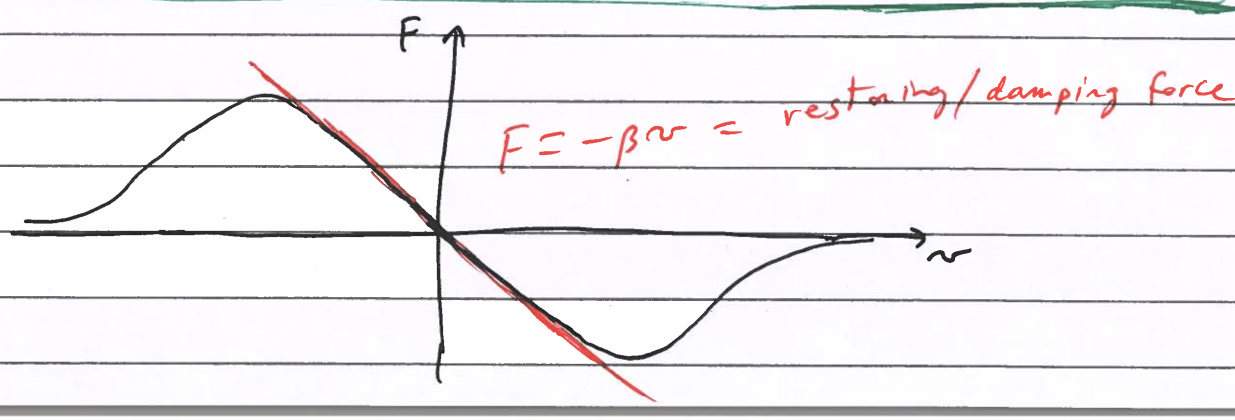
$$\simeq \frac{\hbar k \gamma s_0}{2}\,\frac{1}{1+s_0+4\delta_\ell^2/\gamma^2}\left\{ \cancel{-1} + \frac{8kv\delta_\ell/\gamma^2}{\underline{1+s_0+4\delta_\ell^2/\gamma^2}} + \cancel{1} + \frac{8kv\delta_\ell/\gamma^2}{\underline{1+s_0+4\delta_\ell^2/\gamma^2}} \right\}$$

$$\simeq \frac{\hbar k \gamma s_0}{\left(1+s_0+4\delta_\ell^2/\gamma^2\right)^2}\,\frac{8kv\delta_\ell}{\gamma^2}$$

$$\Longleftrightarrow \boxed{F \simeq \underbrace{\frac{8\hbar k^2 \delta_\ell s_0}{\gamma\left(1+s_0+4\delta_\ell^2/\gamma^2\right)^2}}_{-\beta} v = -\beta v}$$

β: damping coefficient

$-\beta$: negative for red detuning ($\omega_\ell < \omega_{eg} \Rightarrow \delta_\ell < 0$)



Doppler Temperature

The final temperature is determined by the equilibrium condition:

$$\text{cooling rate} = \text{heating rate}$$

What's the cooling rate?

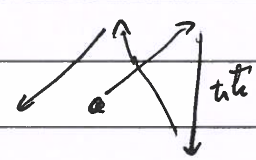
$$\text{Cooling rate} = \frac{dE}{dt} = \text{power out}$$

$$= \vec{F}_{cooling} \cdot \vec{v}$$

$$= \frac{s_0}{1+s_0+\left(\frac{2\delta_\ell}{\gamma}\right)^2} \, 8\hbar k^2 \delta_\ell \, v^2$$

What's the heating rate?

Random direction of absorption (3D) and random fluorescence scattering generates a random walk in momentum space with a step size of $\underbrace{\vec{p} = \hbar \vec{k}}_{\text{absorption or re-emission in 3D}}$



$$\text{heating rate} = \frac{d}{dt}\langle E_{kinetic}\rangle = \frac{d}{dt}\left[\left\langle \frac{p^2(t)}{2m}\right\rangle\right]$$

$$= \frac{1}{2m}\frac{d}{dt}\left\langle\left(\vec{p}_1+\vec{p}_2+\vec{p}_3+\dots\right)^2\right\rangle_t$$

$$= \frac{1}{2m}\frac{d}{dt}\left\langle \vec{P}_1^{\,2} + \vec{P}_2^{\,2} + \dots + 2\vec{P}_1\cdot\vec{P}_2 + 2\vec{P}_2\cdot\vec{P}_3 + \dots \right\rangle_t$$

$$= \frac{1}{2m}\frac{d}{dt}\Big\{ \underbrace{\langle \vec{P}_1^{\,2}\rangle_t}_{(\hbar k)^2} + \underbrace{\langle \vec{P}_2^{\,2}\rangle_t}_{(\hbar k)^2} + \dots + \underbrace{2\cancel{\langle \vec{P}_1\cdot\vec{P}_2\rangle_t}^{\,0} + 2\cancel{\langle \vec{P}_2\cdot\vec{P}_3\rangle_t}^{\,0} + \dots}_{\text{no correlation between different scattering events}} \Big\}$$

no correlation between different scattering events

$$\langle \vec{P}_i\cdot\vec{P}_j\rangle_t = 0 \quad i \neq j$$

$$= \frac{1}{2m}\frac{d}{dt}\Big\{ N (\hbar k)^2 \Big\}$$

$N(t)$

$$= \frac{1}{2m}(\hbar k)^2 \underbrace{\frac{d}{dt}N(t)}_{2\times\gamma_{scattering}}$$

$1+1 =$ absorption + spontaneous emission

$$\Rightarrow \text{heating rate} = \frac{1}{2m}\,2(\hbar k)^2\,\underbrace{\gamma_{scattering}}_{2\times\frac{s_0\,\gamma/2}{1+s_0+\left(\frac{2\delta_\ell}{\gamma}\right)^2}}$$

2 laser beams counterpropagating

for $\langle v\rangle = 0$ ⇐ no Doppler shift on average (neglect)

<u>at equilibrium</u>, heating rate = cooling rate

$$\frac{2(\cancel{\hbar}k)^2}{2m}\,\cancel{\frac{\cancel{s_0}\,\gamma}{1+s_0+\left(\frac{2\delta_\ell}{\gamma}\right)^2}} = \frac{\cancel{s_0}}{\left[1+s_0+\left(\frac{2\delta_\ell}{\gamma}\right)^2\right]^{\cancel{2}}}\,\frac{8\hbar k^2}{\gamma}\,\delta_\ell\, v^2$$

$$\Leftrightarrow \quad \frac{2\hbar\gamma}{2m} = \frac{8\,\delta_\ell\, v^2}{\gamma\left[1+\cancel{s_0}+\left(\frac{2\delta_\ell}{\gamma}\right)^2\right]}$$

stopped here $\qquad \simeq 0$ for low intensity limit (neglect)

$$\Leftrightarrow \quad \underbrace{\frac{\hbar\gamma}{2\cdot 8}\,\frac{\gamma}{\delta_\ell}\left[1+\left(\frac{2\delta_\ell}{\gamma}\right)^2\right]} = \frac{1}{2}mv^2 = \langle E_{kinetic}\rangle$$

This function has a minimum for $\delta_\ell = -\gamma/2$

$$\Rightarrow \quad \frac{\hbar\gamma}{\cancel{2}\cdot 8}\,\frac{\cancel{\gamma}}{\cancel{\gamma}/\cancel{2}}\left(1+\left(\frac{\cancel{2}\,\cancel{\gamma}/\cancel{2}}{\cancel{\gamma}}\right)^2\right) = \underbrace{\langle E_{kinetic}\rangle}_{\frac{1}{2}k_BT}$$

$$\Leftrightarrow \quad \frac{\hbar\gamma}{4} = \frac{1}{2}k_BT_{minimum}$$

$$\Rightarrow \quad T_{minimum} = \frac{\hbar\gamma}{2k_B} = \text{Doppler cooling limit}$$

or

Doppler Temperature

Ex: $T_{Doppler} \simeq 180\ \mu K$ for ^{87}Rb

with $\hbar = 1.054\times 10^{-34}$ J.s

$\gamma = 2\pi \times 6$ MHz

$k_B = 1.38\times 10^{-23}$ J/K