Quantization of the free EM field (no charges)

(i.e. the QM description of light, i.e. photons)

(I) Wave Equation for Vector Potential $\vec{A}$

In the Coulomb gauge ($\vec{\nabla}\cdot\vec{A}=0$), $\vec{A}$ satisfies the following wave equation (no charges present):

$$\boxed{\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}=0} \quad \text{and} \quad V=0$$

recall:

$$\begin{cases} \vec{E}=-\vec{\nabla}V-\frac{\partial}{\partial t}A=-\frac{\partial\vec{A}}{\partial t} \\ \vec{B}=\vec{\nabla}\times\vec{A} \end{cases}$$

(II) Basic Quantization strategy

- Hamiltonian should be linear in the photon number
  - ↳ $H\sim\hbar\omega N$ → looks like a harmonic oscillator

    number of photons

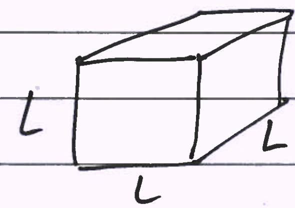
Reminder: $H_{\text{harmonic oscillator}}=\hbar\omega(\hat{N}+\frac{1}{2})=\hbar\omega(\hat{a}^{+}\hat{a}+\frac{1}{2})=\frac{\hbar\omega}{2}(\hat{a}^{+}\hat{a}+\hat{a}\hat{a}^{+})$

- We will use $\vec{A}$ because ~~then~~ then there is only one field to worry about.
- In the basic (original) quantum treatment of photons (by Einstein), the quantities of interest are frequency ω and wavevector $\vec{k}$ (i.e. momentum) of the light, NOT t & x.

decompose $\vec{A}$ in a plane wave basis

III Fourier Expansion of the classical EM field

Consider the universe as a large box with side L and volume V :



We consider running wave solutions $\Rightarrow$ the classically permitted modes are quantized.

[periodic b.c. : $\vec{A}(x=0) = \vec{A}(x=L)$

[NOT quantum quantized]

The plane wave solutions for $\vec{A}$ are

$$\vec{A}_{k,s} = A_{k,s}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\, \hat{\varepsilon}_{k,s} + \underbrace{c.c.}_{A^*_{k,s}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\, \hat{\varepsilon}_{k,s}}$$

note: $\hat{\varepsilon}_{k,s}$ define the $\underbrace{\text{polarization direction}}_{\text{(linear or circular basis)}}$ with: $\begin{cases} \hat{\varepsilon}_{k,s}\cdot\vec{k} = 0 \\ \hat{\varepsilon}_{k,s=1}\cdot\hat{\varepsilon}_{k,s=2} = 0 \end{cases}$

Here, $\omega_k = c\,k$ and $k_{x,y,z} = \frac{2\pi}{L}\, n_{x,y,z}$ with $n_{x,y,z} = 0, \pm 1, \pm 2, \ldots$

Thus, the general solution is

$$\vec{A}(\vec{r},t) = \sum_{k,\, s=1,2} \vec{A}_{k,s}$$

$$= \sum_{k,\, s=1,2} \hat{\varepsilon}_{k,s} \left\{ A_{k,s}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + A^*_{k,s}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} \right\}$$

Amplitude coefficients for plane wave states

Thus

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \sum_{\vec{k}, s=1,2} \hat{\varepsilon}_{\vec{k},s} \left\{ i\omega_k A_{\vec{k},s} e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - i\omega_k A^*_{\vec{k},s} e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} \right\}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \sum_{\vec{k}, s=1,2} i\vec{k} \times \hat{\varepsilon}_{\vec{k},s} \left\{ A_{\vec{k},s} e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + A^*_{\vec{k},s} e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} \right\}$$

IV Classical Hamiltonian

The EM field energy and thus Hamiltonian is given by

(some steps)

$$H \equiv E = \frac{1}{2} \int dV \left\{ \varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 \right\}$$

After some math (substitute $\vec{E}$ & $\vec{B}$) and lots of Algebra, we get ...

$$\left[\text{also: } \int_V dv\, e^{\pm i(\vec{k}-\vec{k}')\cdot\vec{r}} = V \delta_{\vec{k},\vec{k}'} \right]$$

$$\boxed{H_{\substack{\text{classical} \\ \text{EM}}} = \varepsilon_0 V \sum_{\vec{k},s} \omega_k^2 \left\{ A_{\vec{k},s} A^*_{\vec{k},s} + A^*_{\vec{k},s} A_{\vec{k},s} \right\}}$$

V Quantization by analogy

$$H_{\substack{\text{quantum} \\ \text{harmonic} \\ \text{oscillator}}} = \frac{1}{2} \hbar \omega \left(\hat{a} \hat{a}^+ + \hat{a}^+ \hat{a} \right)$$

for many independent harmonic oscillators: $H_{\text{quantum}} = \sum_i \frac{1}{2} \hbar \omega_i \left(\hat{a}_i \hat{a}_i^+ + \hat{a}_i^+ \hat{a}_i \right)$

We postulate that the Quantum Hamiltonian for the free EM field is

$$\boxed{\begin{aligned} H_{\substack{\text{photon}\\ \text{field}}} &\equiv \sum_{k,s} \frac{1}{2}\hbar\,\omega_{k,s}\left\{\hat{a}_{k,s}\hat{a}^{+}_{k,s} + \hat{a}^{+}_{k,s}\hat{a}_{k,s}\right\} \\ \text{with } A_{k,s} &\equiv \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\,\hat{a}_{k,s}\,,\quad A^{*}_{k,s} = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\,\hat{a}^{+}_{k,s}\end{aligned}}$$

VI Quantum states of Light

A. Fock states basis

The eigenstates of the harmonic oscillator are the photon occupation states:

$$|\psi\rangle = |n_{k,s}\rangle = n \text{ photons in the mode with wavevector } \underline{k} \text{ and polarization } \underline{s}.$$

"no photons" is still a state $|0_{k,s}\rangle$ with energy $\frac{\hbar\omega_k}{2}$
↳ called the vacuum state.

More generally, multiple photon states can be simultaneously occupied:

$$|\psi\rangle = |\{n\}\rangle = \underbrace{|n_{k_1,\uparrow}\rangle|n_{k_2,\rightarrow}\rangle \cdots |n_{k_m,\uparrow}\rangle|n_{k_m,\rightarrow}\rangle}_{\text{order does not matter}}$$

most general

$$|\psi\rangle = \sum_{(k_1,s_1),\ldots,(k_m,s_m)} C_{k_1 s_1 \ldots k_m s_m}\, |n_{k_1,s_1}\rangle \cdots |n_{k_m,s_m}\rangle$$