Thursday, December 4, 2025

Reminder from last time: the Quantum Hamiltonian for the free EM field is

$$\boxed{\begin{aligned} H_{\substack{\text{photon}\\ \text{field}}} &\equiv \sum_{k,s} \frac{1}{2}\hbar\omega_{k,s}\left\{\hat{a}_{k,s}\hat{a}^{+}_{k,s} + \hat{a}^{+}_{k,s}\hat{a}_{k,s}\right\} \\ \text{with } A_{k,s} &\equiv \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\,\hat{a}_{k,s}, \quad A^{*}_{k,s} \equiv \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\,\hat{a}^{+}_{k,s}\end{aligned}}$$

VI. Quantum states of Light

A. Fock states basis

The eigenstates of the harmonic oscillator are the photon occupation states:

$$|\psi\rangle = |n_{k,s}\rangle = n \text{ photons in the mode with wavevector } \underline{k} \text{ and polarization } \underline{s}.$$

$\underbrace{\text{"no photons"}}_{\text{called the vacuum state.}}$ is still a state $|0_{k,s}\rangle$ with energy $\frac{\hbar\omega_k}{2}$

More generally, multiple photon states can be simultaneously occupied:

$$|\psi\rangle = |\{n\}\rangle = \underbrace{|n_{k_1,\uparrow}\rangle|n_{k_2,\rightarrow}\rangle \dots |n_{k_m,\uparrow}\rangle|n_{k_m,\rightarrow}\rangle}_{\text{order does not matter}}$$

most general

$$|\psi\rangle = \sum_{(k_1,s_1),\dots,(k_m,s_m)} C_{k_1,s_1,\dots k_m,s_m} |n_{k_1,s_1}\rangle \dots |n_{k_m,s_m}\rangle$$

<u>Review</u>: $$\hat{a}_{k,s}|n_{k,s}\rangle = \sqrt{n_{k,s}}\,|n_{k,s}-1\rangle$$

$$\hat{a}^{\dagger}_{k,s}|n_{k,s}\rangle = \sqrt{n_{k,s}+1}\,|n_{k,s}+1\rangle$$

$$\hat{N}_{k,s}|n_{k,s}\rangle = \underbrace{\hat{a}^{\dagger}_{k,s}\hat{a}_{k,s}}_{=\hat{N}_{k,s}}|n_{k,s}\rangle = n_{k,s}|n_{k,s}\rangle$$

reminder: $M^{\dagger}$ = conjugate transpose of $M = (M^{t})^{*}$

for a Hermitian operator $H^{\dagger} = (H^{t})^{*} = H$

However, $\hat{a}$, $\hat{a}^{\dagger}$ are not Hermitian, so $\hat{a} \neq \hat{a}^{\dagger}$

<u>Commutation Relations</u>:

$$[\hat{a}_{k,s}, \hat{a}^{\dagger}_{k',s'}] = \delta_{k,k'}\delta_{s,s'}$$

$$[\hat{a}_{k,s}, \hat{a}_{k',s'}] = 0$$

$$[\hat{a}^{\dagger}_{k,s}, \hat{a}^{\dagger}_{k',s'}] = 0$$

We can write any Fock state as

$$|\{n\}\rangle = \prod_{k,s} \frac{\left(\hat{a}^{\dagger}_{k,s}\right)^{n_{k,s}}}{\sqrt{n_{k,s}!}}|0\rangle$$

The Fock states form a complete orthonormal basis for the electromagnetic field.

<u>Q</u>: What is a photon?

↳ <u>A</u>: A photon is an excitation of the quantum electromagnetic "potential". Hamiltonian

What the Electric field of a Fock state photon?

Heisenberg representation

$${}_{k,s}\langle n|\vec{E}|n\rangle_{k,s} =$$

$${}_{k,s}\langle n| \sum_{k',s'} \varepsilon_{k',s'} \left\{ i\omega_{k'} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{k'} V}}\, \hat{a}_{k',s'}\, e^{i(k'\cdot r - \omega_{k'} t)} - i\omega_{k'} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{k'} V}}\, \hat{a}^{+}_{k',s'}\, e^{-i(\vec{k}'\cdot\vec{r} - \omega_{k'} t)} \right\} |n\rangle_{k,s}$$

unit vector (polarization)

$${}_{k,s}\langle n|\hat{a}_{k',s'}|n\rangle_{k,s} \qquad\qquad {}_{k,s}\langle n|\hat{a}^{+}_{k',s'}|n\rangle_{k,s}$$

$$= {}_{k,s}\langle n|\, \delta_{k,k'}\, \delta_{s,s'} \sqrt{n_{k,s}}\, |n_{k,s}-1\rangle \qquad = {}_{k,s}\langle n|\, \delta_{k,k'}\, \delta_{s,s'} \sqrt{n_{k,s}+1}\, |n+1\rangle_{k,s}$$

$$= 0 \qquad\qquad = 0$$

so $\boxed{{}_{k,s}\langle n|\vec{E}|n\rangle_{k,s} = 0}$ (not time averaged!!)

What's the variance ΔE of the electric field?

$$\Delta E^2 = \langle n|E^2|n\rangle - \big(\underbrace{\langle n|E|n\rangle}_{=0}\big)^2 \qquad \hat{=} \text{ Intensity in this case}$$

$$= {}_{k,s}\langle n| \left\{ \sum_{k',s'} \left(cst\; \hat{a}_{k',s'} + cst^{*}\; \hat{a}^{+}_{k',s'} \right\}^2 |n\rangle_{k,s}$$

The only possible non-zero terms are

$$\langle n|\,|cst|^2 \underbrace{\hat{a}_{k,s}\hat{a}^{+}_{k,s}}_{\sqrt{n+1}\,\sqrt{n+1}} + |cst|^2 \underbrace{\hat{a}^{+}_{k,s}\hat{a}_{k,s}}_{\sqrt{n}\,\sqrt{n}} + (cst)^2 \underbrace{\hat{a}_{k,s}\hat{a}_{k,s}}_{\langle n|\sqrt{n-1}\,\sqrt{n}\,|n-2\rangle = 0} + (cst^*)^2 \underbrace{\hat{a}^{+}_{k,s}\hat{a}^{+}_{k,s}}_{\langle n|\sqrt{n+2}\,\sqrt{n+1}\,|n+2\rangle = 0}\,|n\rangle$$

$$= \langle n|\,|cst|^2(n+1) + |cst|^2(n)\,|n\rangle$$

$$= |cst|^2\,(2n+1)$$

Thus

$$\boxed{\langle n|\,\Delta E^2\,|n\rangle = \omega_k^2\left(\frac{\hbar}{2\varepsilon_0\omega_k V}\right)(2n+1) = \frac{\hbar\omega_k}{2\varepsilon_0 V}\,(2n+1)}$$

even for $n=0$, there are fluctuations (i.e. vacuum fluctuations)

$\Rightarrow$ Variance (and intensity) is non-zero

$\hookrightarrow$ This approach indicates that $|\vec{E}| = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\sqrt{2n+1}$

Paradox ?

Q: How can the E-field be zero, but the electric field noise (or intensity) be non-zero?

$\hookrightarrow$ A: A Fock state is a quantum state of light with no classical analog.

The $|n\rangle$ state produces no time development of $\vec{E}$!
↳ i.e. there is no phase associated with $\vec{E}$!

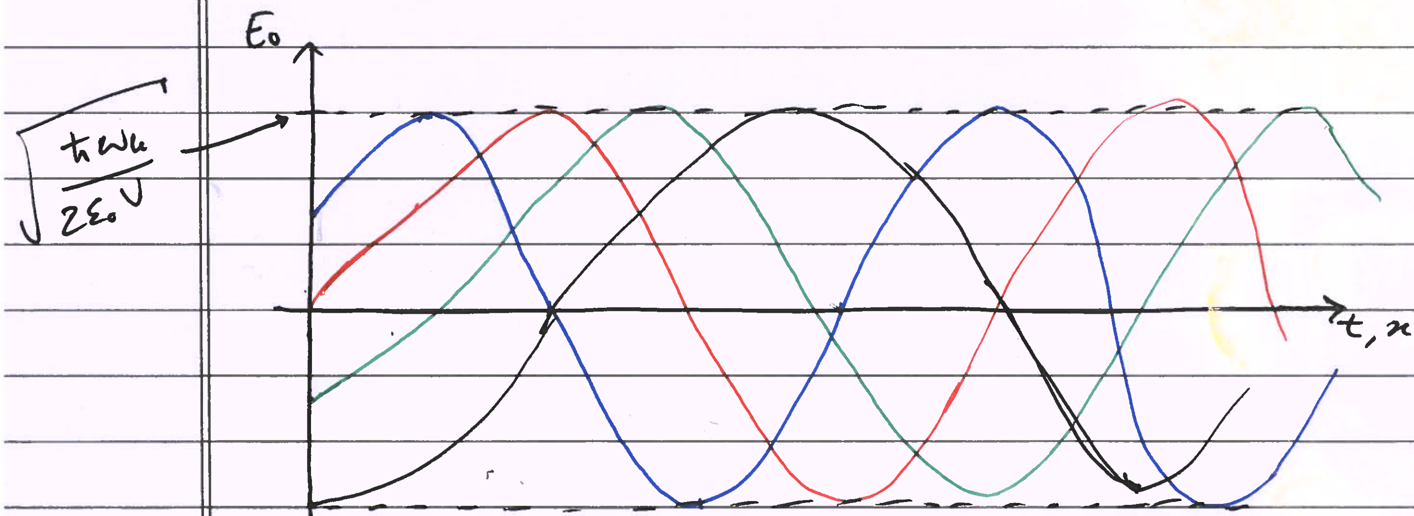
However, photon number $\underline{\underline{n}}$ is completely well defined:

$$\langle n|\hat{N}|n\rangle = n$$

$$\langle n|\Delta N^2|n\rangle = \langle n|\hat{N}^2|n\rangle - \left(\langle n|\hat{N}|n\rangle\right)^2 = n^2 - n^2 = 0$$

In practice, it is difficult (not impossible) to make a Fock state $|n\rangle$ in the lab, except for $|0\rangle$.

Physical picture: Electric field is a superposition of many phase states.



⇒ - If you measure $\vec{E}$ (even for $n=0$), then you will get a non-zero result

- $\langle n|\vec{E}|n\rangle = 0$, i.e. multiple measurements average to zero.
- $\langle n|\Delta\vec{E}^2|n\rangle \neq 0$.
- Amplitude is well defined.
- Phase is completely undefined.

Coherent states

Classical states of light (e.g. laser light) are associated with coherent states:

Definition: coherent state $|\alpha\rangle$ (α = complex number)

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

normalized $\langle\alpha|\alpha\rangle = 1$

A coherent state is an eigenstate of the annihilation operator

$$\boxed{\begin{aligned} \hat{a}|\alpha\rangle &= \alpha|\alpha\rangle \\ \langle\alpha|\hat{a}^+ &= \langle\alpha|\alpha^* \end{aligned}}$$

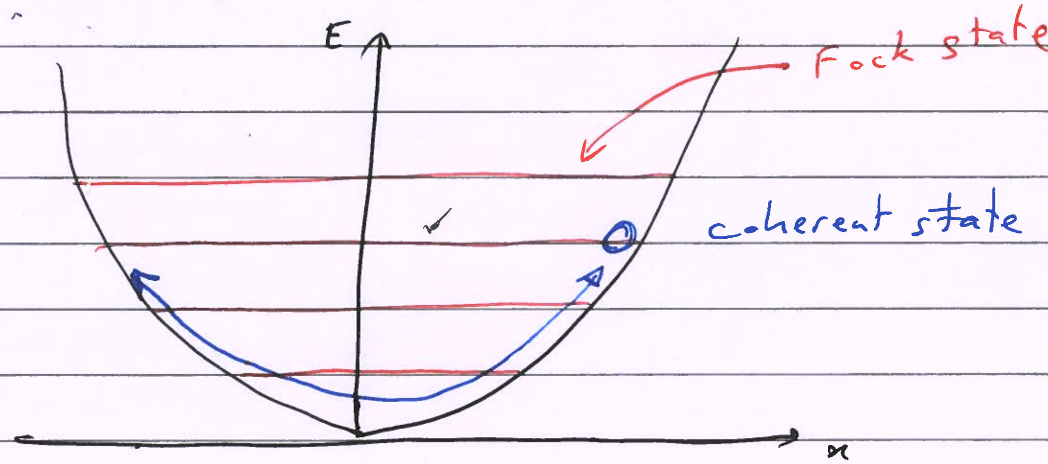
non-Hermitian

Coherent states are generally not orthogonal to each other.

do NOT form a basis

Note: $\langle\alpha|\beta\rangle \neq 0$, actually $|\langle\alpha|\beta\rangle|^2 = \exp\left(-|\alpha-\beta|^2\right)$

A coherent state of a harmonic oscillator represents the probability amplitude (i.e. wavefunction) for a classical particle oscillating in a harmonic potential.



A coherent state also obeys the minimum Heisenberg uncertainty relation $\Delta X \, \Delta P = \frac{\hbar}{2}$

properties: $\langle n \rangle = {}_{k,s}\langle \alpha | \hat{N}_{k,s} | \alpha \rangle_{k,s}$

$$= {}_{k,s}\langle \alpha | \hat{a}^{+}_{k,s} \hat{a}_{k,s} | \alpha \rangle_{k,s} = |\alpha|^2$$

Also, $\Delta N^2 = {}_{k,s}\langle \alpha | \hat{N}^2_{k,s} | \alpha \rangle_{k,s} - \left({}_{k,s}\langle \alpha | \hat{N}_{k,s} | \alpha \rangle_{k,s} \right)^2$

$$a^{+}a\,a^{+}a = a^{+}(a^{+}a+1)a = a^{+}a^{+}aa + a^{+}a$$

$$= \alpha^{*}\alpha^{*}\alpha\alpha + \alpha^{*}\alpha - \langle n \rangle^2$$

$$= |\alpha|^4 + |\alpha|^2 - \langle n \rangle^2$$

$$= \langle n \rangle^2 + \langle n \rangle - \langle n \rangle^2$$

$$= \langle n \rangle$$

Thus $\langle (\Delta N)^2 \rangle = \langle n \rangle \Rightarrow \langle \Delta N \rangle = \sqrt{\langle n \rangle}$

↳ coherent states have $\langle n \rangle = |\alpha|^2$ photons on average, and they have a spread of $\pm\sqrt{\langle n \rangle} = \pm|\alpha|$ photons

↳ coherent states obey Poisson statistics

↳ The photon number can fluctuate!

↳ it is not fixed (as in the Fock state)