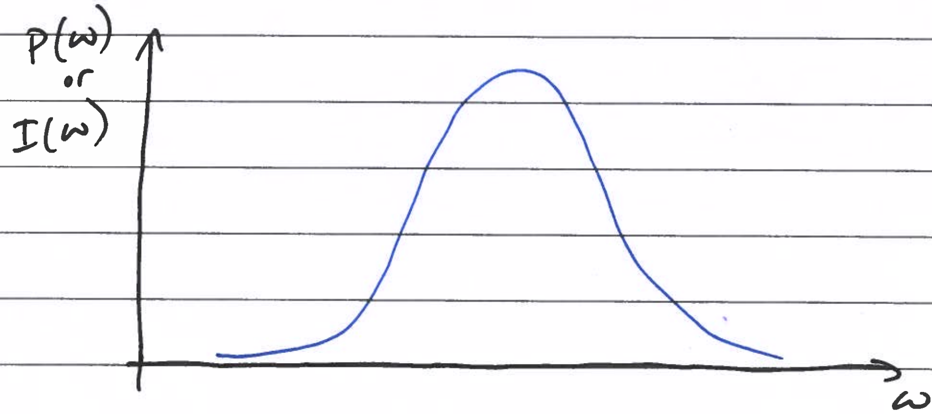
Tuesday, September 9, 2025 recall: $g^{(1)}(\tau) = \frac{\langle E^*(t) E(t+\tau)\rangle_t}{\langle E^*(t) E(t)\rangle_t}$ #1

The Wiener – Khintchine Theorem

Consider a light field with the following spectrum $P(\omega)$



Objective: We would like to find a relationship between "$P(\omega)$" and $g^{(1)}(\tau)$.

The amplitude of the electric field distribution is given by

$$\underbrace{E_{\Delta T}(\omega) = \frac{1}{\sqrt{2\pi}} \int_T^{T+\Delta T} E(t)\, e^{i\omega t}\, dt}_{\text{Fourier transform}} \quad \text{where } \Delta T \gg \tau_c$$

In reverse:

$$E(t) = \frac{1}{\sqrt{2\pi}} \int_\omega^{\omega+\Delta\omega} E_{\Delta T}(\omega)\, e^{-i\omega t}\, d\omega \quad \text{with } \Delta\omega \gg \text{ than the spectral width}$$

the power spectral density $P(\omega)$ is defined as

multiply by "$\varepsilon_0 c$" to get power/intensity

$$P(\omega) = J(\omega) = \frac{1}{\Delta T} \left| E_{\Delta T}(\omega) \right|^2$$

average over t

$$= \frac{1}{\Delta T} \frac{1}{(\sqrt{2\pi})^2} \int_{\Delta T} dt \int_{\Delta T} dt'\, E^*(t) E(t') \overbrace{e^{i\omega(t'-t)}}^{e^{i\omega\tau}}$$

substitution: $\begin{cases} \tau = t' - t \\ d\tau = dt' \end{cases}$

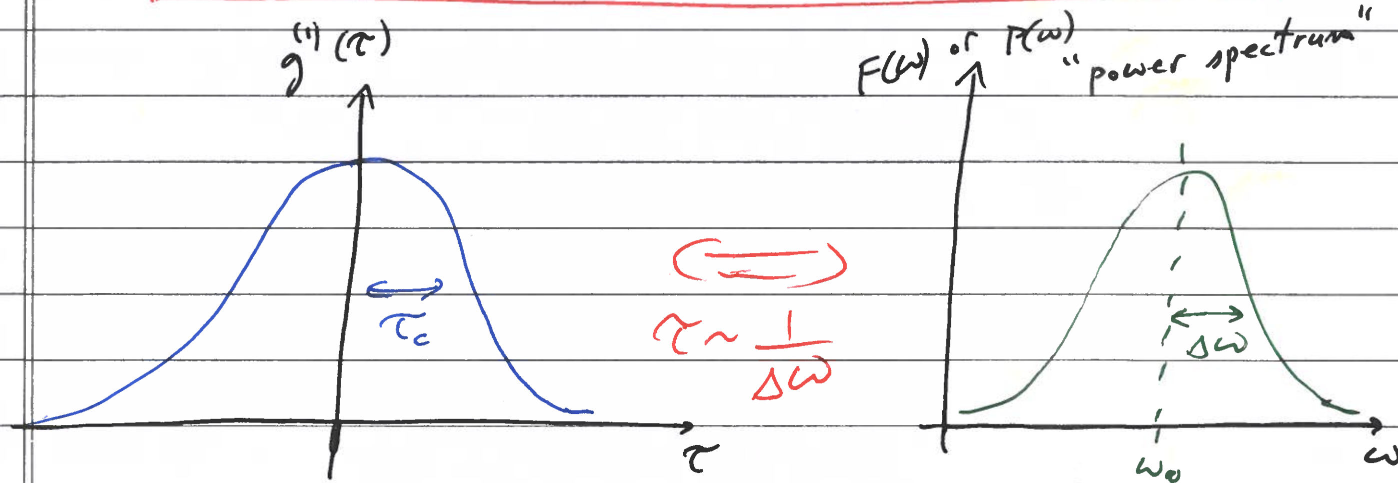
$$= \frac{1}{2\pi} \int_{\Delta T} d\tau \, \langle E^*(t)\, E(t+\tau) \rangle_t \, e^{i\omega\tau}$$

define normalization: $\mathcal{N} = \int_{-\infty}^{+\infty} d\omega \, f(\omega) = \langle E^*(t)\, E(t) \rangle$

Thus $$\boxed{F(\omega) = \frac{f(\omega)}{\mathcal{N}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g^{(1)}(\tau)\, e^{i\omega\tau}\, d\tau}$$

normalized power spectral density

Wiener-Khintchine Theorem



$g^{(1)}(\tau)$ measures the spectral width of the light.

A Michelson interferometer measures $|g^{(1)}(\tau)|$ and the spectral width of the light.

2^{nd} order auto-correlation function

definition: $$g^{(2)}(\tau) = \frac{\langle I(t) \cdot I(t+\tau) \rangle_t}{\langle I(t) \rangle_t^2}$$

Example: Monochromatic light (plane wave $\simeq$ laser light)

$$E(t) = E_0 \, e^{i(kz - \omega t + \phi)}$$
$$\equiv E_0 \cos(kz - \omega t + \phi)$$

at $z = 0$, $I(t) = \varepsilon_0 c E_0^2 \underbrace{\cos^2(\omega t - \phi)}_{1/2 \text{ on average}}$

$$\Rightarrow \underbrace{\langle I(t) \rangle_t}_{\substack{\text{average over} \\ \text{many optical cycles}}} = \frac{1}{2} \varepsilon_0 c E_0^2$$

$$\langle I(t) \cdot I(t+\tau) \rangle = (\varepsilon_0 c)^2 E_0^4 \underbrace{\langle \cos^2(\omega t - \phi) \cos^2(\omega(t+\tau) - \phi) \rangle_t}$$

$$\left\langle \left\{ \frac{1}{2} \left[\cos(2\omega t + \omega\tau - 2\phi) + \cos(\omega\tau) \right] \right\}^2 \right\rangle_t$$

$$= \frac{1}{4} \Big[\underbrace{\langle \cos^2(\;) \rangle_t}_{\frac{1}{2}} + \underbrace{\langle \cos^2(\;) \rangle_t}_{\frac{1}{2}} + \underbrace{\langle 2 \cos(\;) \cos(\;) \rangle_t}_{0} \Big]$$

note: our photodetector is not fast enough to see changes at frequency ω or 2ω ... Also, average over small τ changes as well.

$$= \frac{1}{4} (\varepsilon_0 c)^2 E_0^4$$

thus $$\boxed{g^{(2)}(\tau) = \frac{\frac{1}{4} (\varepsilon_0 c)^2 E_0^4}{\left[\frac{1}{2} (\varepsilon_0 c) E_0^2 \right]^2} = 1}$$ for all τ