Physics 430 and 631: Quantum Optics & Atomics
Due date: Thursday, September 18, 2025.

Problem Set #3

1. Simple thermal light model: monochromatic waves with random phases

Consider an ensemble of N atoms, which each emit scalar monochromatic waves of frequency $\omega_j = \omega$ with time-dependent random phases $\phi_j(t)$. There are no correlations between the random phases of the atoms. For simplicity, we will assume that the amplitudes of all these waves are the same (i.e., E_0), when measured at a far away detector.

The total electric field at some detector far from the atoms is then given by:

$$E_{total}(t) = \sum_{j=1}^{N} E_j(t) = E_0 e^{-i\omega t} \sum_{j=1}^{N} e^{i\phi_j(t)}$$

a) Show that $\langle I_{total}(t) \rangle = \frac{1}{2}\varepsilon_0 c E_0^2 N = \langle I_{single\ atom} \rangle N$.

Here c is the speed of light, ε_0 is the permittivity of free space, and $\langle \cdot \rangle$ represents an average over the time variable t (i.e., an average over many optical cycles).

b) Calculate the 2nd order auto-correlation function $g^{(2)}(\tau = 0)$ and show that in the large N limit, $g^{(2)}(0) = 2$.

Note: This calculation is somewhat lengthy (1-2 pages).

c) Explain why or derive $g^{(2)}(\tau \gg \tau_c) = 1$, where τ_c is the coherence time, i.e., the characteristic time scale on which the random phases $\phi_j(t)$ vary.

2. Level crossings

Consider a 2-level atom with ground and excited states $|g\rangle$ and $|e\rangle$, respectively, and energies, E_g and E_e, respectively. These energies depend on a parameter *m* such that

$$E_g(m) = E_g(0) + \alpha\cdot m \text{ and } E_e(m) = E_e(0) - \alpha\cdot m$$

We modify the basic Hamiltonian of the system, H_0, by adding a generic interaction with Hamiltonian:

$$H_{\text{int}} = \begin{bmatrix} 0 & W \\ W^* & 0 \end{bmatrix}$$

a) Calculate the new eigenenergies of the system with the interaction present, as a function of m.

b) Plot the energy of the system as function of *m*, with and without the interaction present.

c) Calculate the new eigenstates of the system with the interaction present, and show that one can go continuously from modified state |g⟩ to modified state |e⟩, and vice versa, by adiabatically varying *m*. What is the condition for adiabaticity? Make a quantitative (and logical) argument. What happens if you ramp *m* much faster than the adiabatic condition (support your answer with a quantitative argument)?

------------------------- *Extra graduate student problem* -------------------------

3. Gaussian lineshape

Consider a gas of identical atoms that emit light at a resonant frequency of ω_0 (when at rest). The atoms in the gas have a spread of velocities due to their finite temperature, T, which leads to a spread in the resonant frequency of the atoms due to the Doppler effect. The probability for an atom to emit light at a frequency ω close to ω_0 in a given direction is then given by

$$P(\omega) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(\omega-\omega_0)^2}{\sigma^2} \right) \quad \text{with} \quad \sigma = \omega_0 \sqrt{\frac{kT}{mc^2}} ,$$

where *m* is the mass of the atom, *c* is the speed of light, and *k* is Boltzmann's constant.

We will assume that the atom is excited by some mechanism, and we look at the radiated light in a given direction. The emitted electric field in this direction is then given by

$$E(t) = E_0 \sum_{i=1}^{N} \exp(-i\omega_i t + i\phi_i)$$

where the sum is over the N atoms in the gas and the ϕ_i are random stationary phases.

a. Show that $\left\langle E^*(t)E(t+\tau) \right\rangle = E_0^2 \sum_{i=1}^{N} \exp(-i\omega_i \tau)$.

b. Show that $g^{(1)}(\tau) = \exp\left(-i\omega_0\tau - \frac{1}{2}\sigma^2\tau^2 \right)$.