Tuesday, September 16, 2025

$\underbrace{\text{2-level atom}}_{\text{QM}}$ in an $\underbrace{\text{EM field}}_{\text{classical EM}}$

(I) Hamiltonian

The interaction between an electric field $\vec{E}$ and the electric dipole moment $\vec{d}$ of an atom is given by

$$H_{int} = -\vec{d}\cdot\vec{E} = e\vec{R}\cdot\vec{E}_0 \underbrace{\cos(\omega_\ell t)}_{\substack{\cos(kz-\omega_\ell t) \\ \text{with } z=0}}$$

(see Griffiths eqn. 4.6)

ω_ℓ = frequency of laser field

(Assumption: single e^- orbiting a nucleus)

In the 2-level atom basis $\{|g\rangle, |e\rangle\}$, we have

$$H_{int} = e\vec{E}_0\cos(\omega_\ell t)\begin{bmatrix} \langle g|\vec{R}|g\rangle & \langle g|\vec{R}|e\rangle \\ \langle e|\vec{R}|g\rangle & \langle e|\vec{R}|e\rangle \end{bmatrix}$$

Parity operator

Recall: $[\mathbb{P}, H_{atom}] = 0 \implies |g\rangle$ & $|e\rangle$ have a definite parity.

"2-level atom" $\simeq$ alkali atom (1 valence e^-)

$nS \longleftrightarrow nP$ transition

$|g\rangle \longleftrightarrow |e\rangle$

$$\langle g|\vec{R}|g\rangle = \langle \underset{\text{even}}{g}|\underset{\text{odd}}{(x,y,z)}|\underset{\text{even}}{g}\rangle = \int \underset{\text{even}}{\psi_g^*(\vec{r})}\,\underset{\text{odd}}{(x,y,z)}\,\underset{\text{even}}{\psi_g(\vec{r})}\,d^3r$$

$$= 0$$

similarly, $\langle e|\vec{R}|e\rangle = 0$

In AMO, we define $\Omega = \dfrac{e\langle g|\vec{R}|e\rangle \cdot \vec{E}_0}{\hbar}$

$$= \frac{\langle g| H_{int} |e\rangle}{\hbar}$$

$$= \text{Rabi frequency}$$

Thus the Hamiltonian for a 2-level atom interacting with an EM field is

$$H = \underbrace{\begin{array}{cc} & \begin{array}{cc} |g\rangle & |e\rangle \end{array} \\ \begin{array}{c} \langle g| \\ \langle e| \end{array} & \begin{pmatrix} E_g & 0 \\ 0 & E_e \end{pmatrix} \end{array}} + \underbrace{\hbar \begin{array}{cc} \begin{array}{cc} |g\rangle & |e\rangle \end{array} \\ \begin{pmatrix} 0 & \Omega \\ \Omega^* & 0 \end{pmatrix} \end{array} \cos(\omega_e t)}_{H_{int}}$$

$$H_0 = \hbar \begin{pmatrix} \omega_g & 0 \\ 0 & \omega_e \end{pmatrix}$$

Time Evolution : Schrodinger Equation

The Schrodinger equation is

$$(1) \quad i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$$

We will find a solution in the unperturbed basis $\{|g\rangle, |e\rangle\}$ of the form

$$(2) \quad |\psi(t)\rangle = \underbrace{C_g(t)}\, e^{-i\omega_g t} |g\rangle + \underbrace{C_e(t)}\, e^{-i\omega_e t} |e\rangle$$

non-trivial time-dependence (due to H_{int})

(2) $\longrightarrow$ (1) "plug & chug"

$|g\rangle$ component:

$$i\hbar\left[\frac{\partial}{\partial t}C_g(t)\cdot e^{-i\omega_g t} + \cancel{C_g(t)\cdot(-i\omega_g)e^{-i\omega_g t}}\right]|g\rangle$$

$$= \left[\cancel{C_g(t)\cdot e^{-i\omega_g t}\underbrace{E_g}_{\hbar\omega_g}} + C_e(t)e^{-i\omega_e t}\,\hbar\Omega\cos(\omega_\ell t)\right]|g\rangle$$

$e^{-i\omega_g t}$

$$\Rightarrow (3a)\quad i\hbar\frac{\partial}{\partial t}C_g(t) = C_e(t)\hbar\Omega\cos(\omega_\ell t)\,e^{-i\underbrace{\omega_{eg}}_{\omega_{eg}=\omega_e-\omega_g}t}$$

similarly,

$$(3b)\quad i\hbar\frac{\partial}{\partial t}C_e(t) = C_g(t)\hbar\Omega^*\cos(\omega_\ell t)\,e^{+i\omega_{eg}t}$$

(no approximation so far!)

Rotating Wave Approximation (RWA)

$$(3a) \longrightarrow i\cancel{\hbar}\frac{\partial}{\partial t}C_g(t) = C_e(t)\cancel{\hbar}\Omega\left[\frac{e^{i\omega_\ell t}+e^{-i\omega_\ell t}}{2}\right]e^{-i\omega_{eg}t}$$

$$\Leftrightarrow i\frac{\partial}{\partial t}C_g(t) = C_e(t)\frac{\Omega}{2}\left[\underbrace{e^{i(\omega_\ell-\omega_{eg})t}}_{\text{slow oscillations}} + \underbrace{e^{-i(\omega_\ell+\omega_{eg})t}}_{\text{very fast oscillations}}\right]$$

$\hookrightarrow$ ignore (RWA)

Partial Justification:

If $C_e(t)\simeq$ cst (slowly varying), then integrate $\dot{C}_g(t)$ (averages to "zero") with respect to t.

$$\Rightarrow i\,C_g(t) \simeq C_e\frac{\Omega}{2}\left[\frac{e^{i(\omega_\ell-\omega_{eg})t}}{i(\omega_\ell-\omega_{eg})} + \underbrace{\frac{e^{-i(\omega_\ell+\omega_{eg})t}}{-i(\omega_\ell+\omega_{eg})}}_{\text{very small}}\right]$$

$$\Rightarrow \begin{cases} i\dfrac{\partial}{\partial t} C_g(t) \simeq C_e(t)\dfrac{\Omega}{2}\, e^{i\overbrace{(\omega_\ell-\omega_{eg})}t} & (4a) \\ \text{similarly,} \\ i\dfrac{\partial}{\partial t} C_e(t) \simeq C_g(t)\dfrac{\Omega^*}{2}\, e^{-i(\omega_\ell-\omega_{eg})t} & (4b) \end{cases}$$

$\delta = \omega_\ell - \omega_{eg}$
$=$ detuning

Rabi Oscillations

We need to get separate equations for $C_g(t)$ & $C_e(t)$

$$\frac{\partial}{\partial t}(4a) \Rightarrow i\frac{\partial^2}{\partial t^2} C_g(t) \simeq C_e(t)\frac{\Omega}{2}(+i\delta)e^{i\delta t} + \frac{\Omega}{2}e^{i\delta t}\underbrace{\left(\frac{\partial}{\partial t}C_e(t)\right)}$$

substitute from (4a) → "$\frac{\partial}{\partial t}C_g(t)$" term

substitute from (4b) "$C_g(t)$" term

(repeat for $\frac{\partial}{\partial t}(4b)$)

After differentiation & substitutions, we get (see Metcalf & Van der Straten)

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}C_g(t) - i\delta\dfrac{\partial}{\partial t}C_g(t) + \dfrac{|\Omega|^2}{4}C_g(t) = 0 & (5a) \\ \dfrac{\partial^2}{\partial t^2}C_e(t) + i\delta\dfrac{\partial}{\partial t}C_e(t) + \dfrac{|\Omega|^2}{4}C_e(t) = 0 & (5b) \end{cases}$$

Easy/straightforward to solve! (but tedious)

(standard 2[nd] order diff. eq. with constant coefficients)

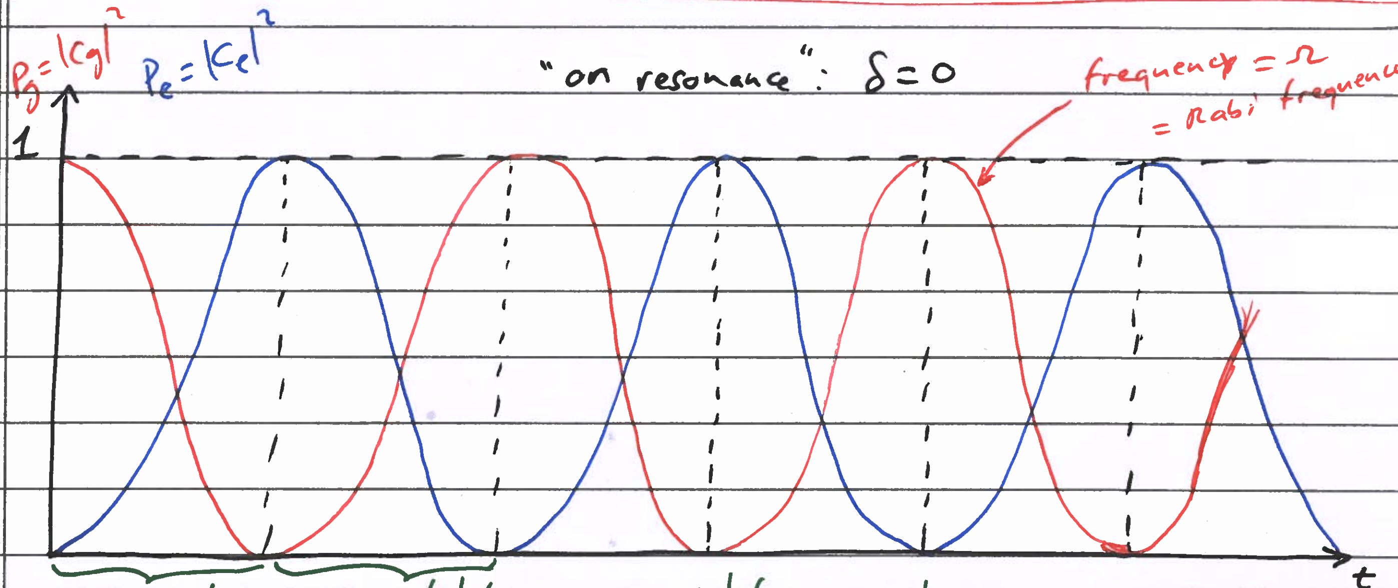
<u>Initial conditions</u>: atom in ground state

$$c_g(t=0)=1 \quad , \quad \frac{\partial}{\partial t} c_g(t=0)=0$$
$$c_e(t=0)=0 \quad , \quad \frac{\partial}{\partial t} c_e(t=0)=0$$

<u>Solution</u>:

$$\begin{cases} c_g(t) = \left[\cos\left(\frac{\Omega' t}{2}\right) - \frac{i\delta}{\Omega'} \sin\left(\frac{\Omega' t}{2}\right)\right] e^{i\frac{\delta}{2}t} & (6a) \\ c_e(t) = -i\frac{\Omega}{\Omega'} \sin\left(\frac{\Omega' t}{2}\right) e^{-i\frac{\delta}{2}t} & (6b) \end{cases}$$

with $\delta = \omega_L - \omega_{eg}$ = detuning and $\Omega' = \sqrt{\Omega^2+\delta^2}$ = generalized Rabi frequency



stimulated absorption — stimulated emission → completely quantum process

For $\delta \gg \Omega$, then $\Omega' \simeq \delta$ = "frequency"
Amplitude $= \left|\frac{\Omega}{\Omega'}\right|^2 \simeq \frac{\Omega^2}{\delta^2}$