Thursday, September 18, 2025 #1

Energy of a 2-level atom in an EM field

The Hamiltonian for the system is time-dependent.

↳ Energy is not conserved, though it is on average.

Let's take a closer look at $c_g(t)$ for $|\delta| \gg |\Omega|$

↳ in this case $\underbrace{|c_g|^2}_{1-\frac{\Omega^2}{\delta^2}} \gg \underbrace{|c_e|^2}_{\frac{\Omega^2}{\delta^2}}$ and $\Omega' \simeq |\delta|$

Thus $|\psi(t)\rangle \simeq c_g(t)\, e^{-i\omega_g t}|g\rangle + \varepsilon|e\rangle$ (ε is small)

$$c_g(t) = \left[\cos\left(\frac{\Omega'}{2}t\right) - i\frac{\delta}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\right] e^{i\frac{\delta}{2}t}$$

sign of δ

$$\simeq e^{-i\frac{\Omega'}{2}t}\, e^{+i\frac{\delta}{2}t}$$

$$\simeq e^{-i\left(\cancel{\frac{\delta}{2}} + \frac{1}{4}\frac{\Omega^2}{\delta} - \cancel{\frac{\delta}{2}}\right)t}$$

$$\simeq e^{-i\frac{1}{4}\frac{\Omega^2}{\delta}t}$$

$$\Omega' = \sqrt{\delta^2+\Omega^2} = |\delta|\sqrt{1+\frac{\Omega^2}{\delta^2}} \simeq |\delta|\left(1+\frac{1}{2}\frac{\Omega^2}{\delta^2}+\dots\right) \simeq |\delta| + \frac{1}{2}\frac{\Omega^2}{|\delta|}+\dots$$

Thus $|\psi(t)\rangle \simeq e^{-i\left(\omega_g + \frac{1}{4}\frac{\Omega^2}{\delta}\right)t}|g\rangle + \varepsilon|e\rangle$

↳ We can interpret this shift in "oscillation frequency" as meaning that the energy of the ground state is shifted

Energy shift of the ground state: $\boxed{E_g' \simeq E_g + \frac{\hbar\Omega^2}{4\delta}}$ for $|\delta| \gg |\Omega|$

↳ Also kind-of-sort-of hints that the system might have "eigenenergies" .... and "eigenstates".

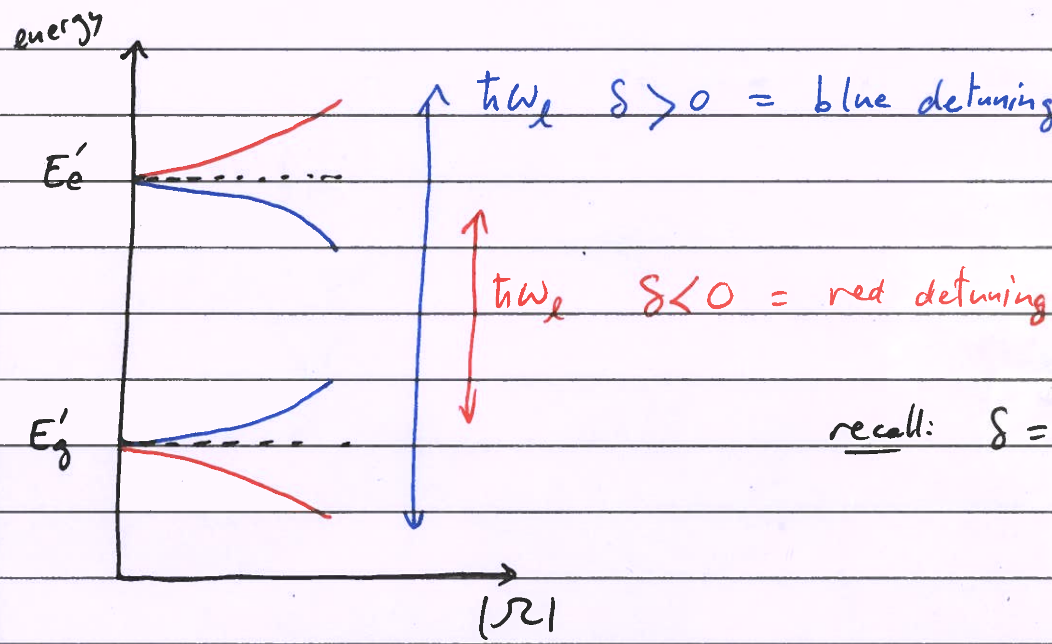
By a similar analysis (with $c_g(0)=0$, $c_e(0)=1$), we find that

$$|\psi(t)\rangle \simeq \varepsilon|g\rangle + e^{-i\left(\omega_e - \frac{1}{4}\frac{\Omega^2}{\delta}\right)t}|e\rangle$$

↳ thus the excited state is shifted: $\boxed{E_e' = E_e - \frac{\hbar\Omega^2}{4\delta}}$

for $|\delta| \gg |\Omega|$

⇒ Energy of the $|e\rangle$ state is shifted exactly opposite to the ground state $|g\rangle$ shift.



recall: $\delta = \omega_\ell - \omega_{eg}$

For $H_{int} = e\vec{E}_0 \cdot \vec{R}\cos(\omega_\ell t)$ ⟶ AC Stark shift

For $H_{int} = -\vec{\mu} \cdot \vec{B}_0 \cos(\omega_\ell t)$ ⟶ AC Zeeman shift

for $\delta = 0$ ⇒ Autler-Townes splitting, Mollow triplet

(our theory cannot do this yet)

Dressed Atom Theory — Another view

Motivation: Include the "quantum photon" description in our 2-level atom system

→ try to conserve ~~tot~~ total energy (atom + photons)

→ try to obtain eigenstates & eigenenergies for system

A_ Interstate Interaction – DC Hamiltonian (time independent)

Consider a 2-level atom with a generic DC interstate interaction W:

$$H = H_0 + H_{int}(W)$$

$$= \begin{array}{c|cc} & |g\rangle & |e\rangle \\ \hline \langle g| & E_g & 0 \\ \langle e| & 0 & E_e \end{array} + \begin{array}{c|cc} & |g\rangle & |e\rangle \\ \hline \langle g| & 0 & W \\ \langle e| & W^* & 0 \end{array}$$

What are the new eigenstates & eigenenergies of H in terms of $|g\rangle, |e\rangle, E_g, E_e,$ and W?

$$\det \begin{pmatrix} E_g - \lambda & W \\ W^* & E_e - \lambda \end{pmatrix} = 0$$

$$\Leftrightarrow (E_g - \lambda)(E_e - \lambda) - |W|^2 = 0$$

If $E_g = -\frac{E_{eg}}{2}$ and $E_e = +\frac{E_{eg}}{2}$, then $\lambda^2 - \left(\frac{E_{eg}}{2}\right)^2 - |W|^2 = 0$

$$\Rightarrow \lambda_\pm = \pm \sqrt{\left(\frac{E_{eg}}{2}\right)^2 + |W|^2}$$

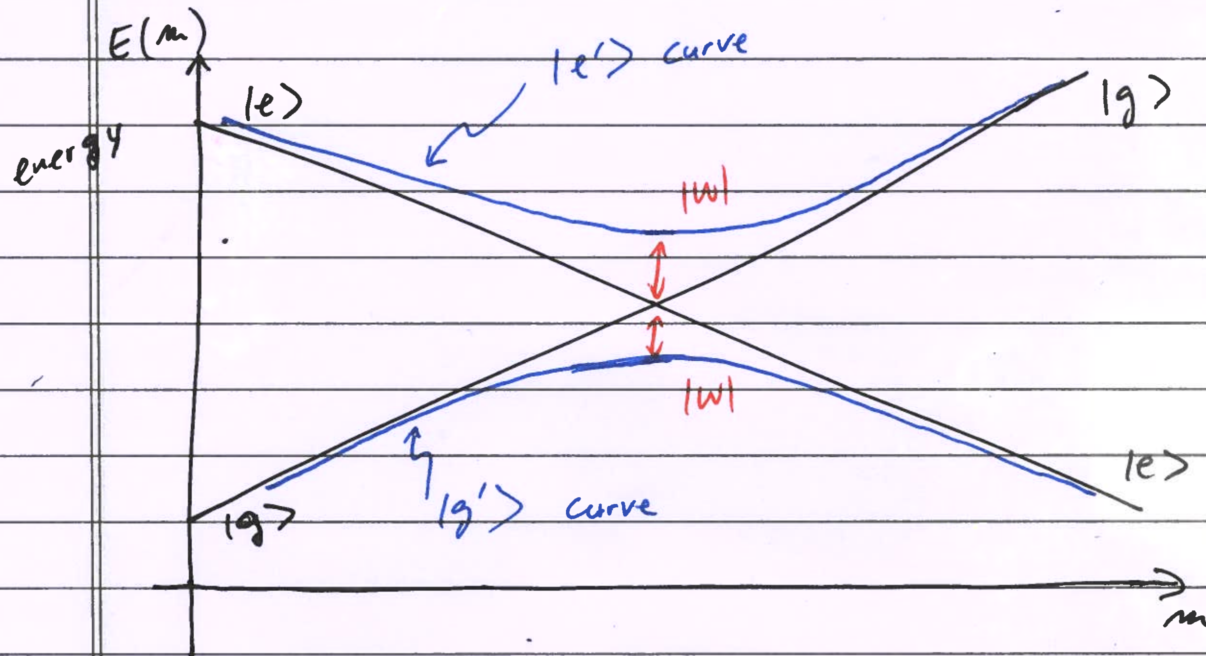
In the new eigenbasis $\{|g'\rangle, |e'\rangle\}$

$$H = \begin{array}{c|cc} & |g'\rangle & |e'\rangle \\ \hline \langle g'| & -\sqrt{\left(\frac{E_{eg}}{2}\right)^2 + |W|^2} & 0 \\ \langle e'| & 0 & +\sqrt{\left(\frac{E_{eg}}{2}\right)^2 + |W|^2} \end{array}$$

the energy eigenstates are repelled.

with $|g'\rangle = |g\rangle + \varepsilon |e\rangle$

$|e'\rangle = |e\rangle - \varepsilon^* |g\rangle$



$\Rightarrow$ If 2 levels cross as a function of some parameter (i.e. m ... e.g, magnetic field, electric field, detuning, etc...), then, if there is any interaction, the energy states/levels will repel.

$\Rightarrow$ avoided level crossing!!

If you ramp $\underline{m}$ adiabatically, then $|g\rangle \to |e\rangle$

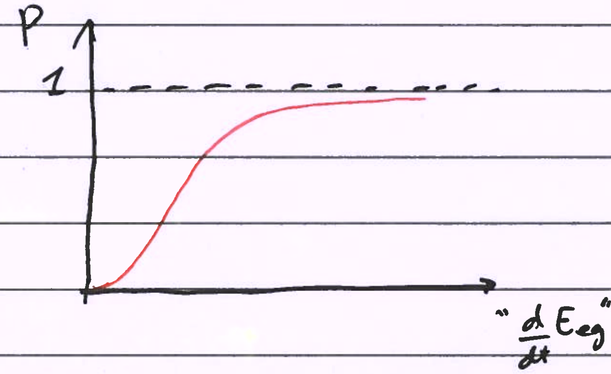
$|e\rangle \to |g\rangle$

$\hookrightarrow$ the system stays on the $|e'\rangle$ or $|g'\rangle$ energy level curve

If you ramp ~~m~~ quickly (diabatically), then you can get a Landau - Zener transition, where you jump from $|g'\rangle$ to $|e'\rangle$ (or vice versa)

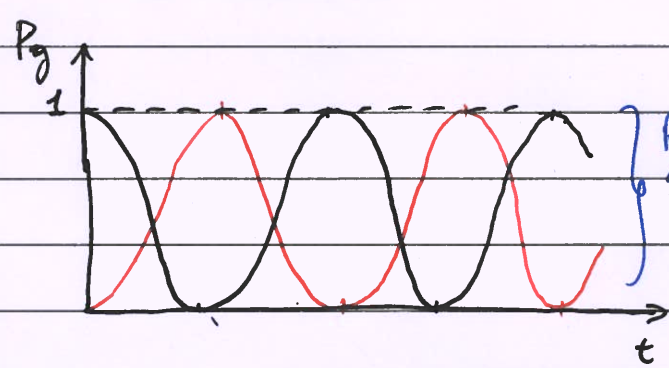
P = Probability to jump curves = $\exp\left[-2\pi\Gamma\right]$

$$\text{with } \Gamma = \frac{|W|^2}{\hbar \cdot \frac{d}{dt} E_{eg}}$$

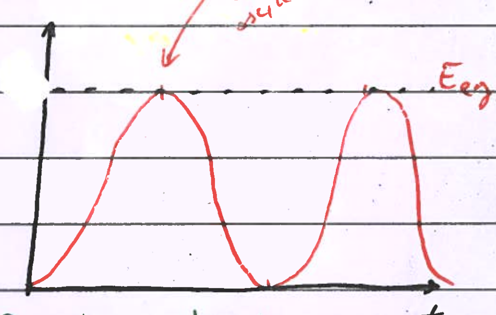


B. Basic Energetics of a 2-level atom in an EM field (laser)

On resonance : $\delta = 0$



$\Longrightarrow$



[omitted interaction energy between the EM field and the atom]

what if we include the energy in the EM field ?

state 1 : $|1\rangle = |g\rangle_a |N+1\rangle_{\omega_{eg}} \rightarrow E_1 = E_g + \hbar\omega_{eg}(N+1)$

state 2 : $|2\rangle = |e\rangle_a |N\rangle_{\omega_{eg}} \rightarrow E_2 = \underbrace{E_e}_{E_g + \hbar\omega_{eg}} + \hbar\omega_{eg} N = E_g + \hbar\omega_{eg}(N+1)$

So $E_1 = E_2 \equiv E_0$, i.e. energy is conserved
(by construction)

State $|1\rangle$ & state $|2\rangle$ are the "bare states" for our dressed atom model.

C. Hamiltonian for a 2-level dressed atom in an EM field
(i.e., how do we make a static system oscillate?)

1st guess

$$\text{"}H\text{"} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$$

However, from the 1st lecture on 2-level atoms (Sept. 11), we know that such a system will not display any oscillations, regardless of which basis we measure in! $(\Delta E = E_2 - E_1 = 0)$

2nd try: In some basis, we would like to write:

$$\text{"}H\text{"} = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 + \underbrace{\hbar\Omega} \end{pmatrix}$$

↳ guarantees an oscillation frequency of Ω in some basis.

from section A, we know that

$$H = \begin{bmatrix} E_0 & \frac{\hbar\Omega}{2} \\ \frac{\hbar\Omega^*}{2} & E_0 \end{bmatrix}$$ will work!

In this approach (dressed atom picture), Rabi flopping occurs because states $|1\rangle$ & $|2\rangle$ (actually, the new eigenstates) no longer have the same energy due to an off-diagonal photon + atom interaction $\frac{\hbar\Omega}{2}$.

STOPPED HERE

Dressed atom Hamiltonian (includes off resonance case)

$$H = H_{atom} + H_{photons} + H_{interaction}$$

$$= \begin{pmatrix} E_g & 0 \\ 0 & E_e \end{pmatrix} + \hbar\omega_\ell \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix} + \hbar \begin{pmatrix} 0 & \Omega/2 \\ \frac{\Omega^*}{2} & 0 \end{pmatrix}$$

in the $\{|1\rangle, |2\rangle\}$ basis, i.e. $\{|g\rangle_a |N+1\rangle_{\omega_\ell}, \; |e\rangle_a |N\rangle_{\omega_\ell}\}$

$$\Rightarrow H = \hbar \begin{pmatrix} \omega_{eg} + \delta & \Omega/2 \\ \Omega^*/2 & \omega_{eg} \end{pmatrix}$$

subtract "$\hbar\omega_\ell N$" & E_g from the diagonal energies ($\delta = \omega_\ell + \omega_{eg}$) [it's just an energy offset]

$$\Longleftrightarrow \boxed{H = \hbar \begin{array}{c|cc} & |1\rangle & |2\rangle \\ \hline \langle 1| & \delta & \Omega/2 \\ \langle 2| & \Omega^*/2 & 0 \end{array}}$$

subtract "$\hbar\omega_{eg}$" from diagonal terms

dressed atom Hamiltonian

(no explicit time dependence!)

<u>Note</u>: A rigorous derivation requires going into the "rotating frame" of the atom/laser (not an approximation).

↳ see future problem set.