Tuesday, September 23, 2025

<u>Dressed atom Hamiltonian</u> (includes off resonance case)

$$H = H_{atom} + H_{photons} + H_{interaction}$$

$$= \begin{pmatrix} E_g & 0 \\ 0 & E_e \end{pmatrix} + \hbar\omega_\ell \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix} + \hbar \begin{pmatrix} 0 & \Omega/2 \\ \Omega^*/2 & 0 \end{pmatrix}$$

in the $\{|1\rangle, |2\rangle\}$ basis, i.e., $\{\underbrace{|g\rangle_a |N+1\rangle_{\omega_\ell}}_{|g, N+1\rangle}, \underbrace{|e\rangle_a |N\rangle_{\omega_\ell}}_{|e, N\rangle}\}$

$$H = \hbar \begin{pmatrix} \omega_{eg} + \delta & \Omega/2 \\ \Omega^*/2 & \omega_{eg} \end{pmatrix}$$

subtract "$E_g + \hbar\omega_\ell N$" from diagonal energies [it's just an energy offset]

reminder: $\delta = \omega_\ell - \omega_{eg}$

$$\Rightarrow H = \hbar \begin{pmatrix} \delta & \Omega/2 \\ \frac{\Omega^*}{2} & 0 \end{pmatrix}$$

subtract "$\hbar\omega_{eg}$" from diagonal terms

<u>Note 1:</u> H is <u>not</u> explicitly time dependent.

<u>Note 2:</u> A rigorous derivation requires going into the rotating frame of the atom (not an approximation).

AC Stark shift

Let's solve for the eigenenergies of the Hamiltonian

$$\det\{H - \lambda I\} = 0 \Leftrightarrow \begin{vmatrix} \delta - \lambda & \Omega/2 \\ \Omega^*/2 & 0 - \lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (\delta - \lambda)(-\lambda) - \frac{|\Omega|^2}{4} = 0$$

$$\Leftrightarrow \lambda^2 - \delta\lambda - \frac{|\Omega|^2}{4} = 0$$

$$\Rightarrow \boxed{\lambda_\pm = \frac{\delta \pm \sqrt{\delta^2 + |\Omega|^2}}{2} = \frac{E_\pm}{\hbar}}$$

for $\delta = 0$: $\boxed{E_\pm = \pm \frac{\hbar |\Omega|}{2}}$. "on resonance"

for $|\delta| \gg |\Omega|$ "far off resonance"

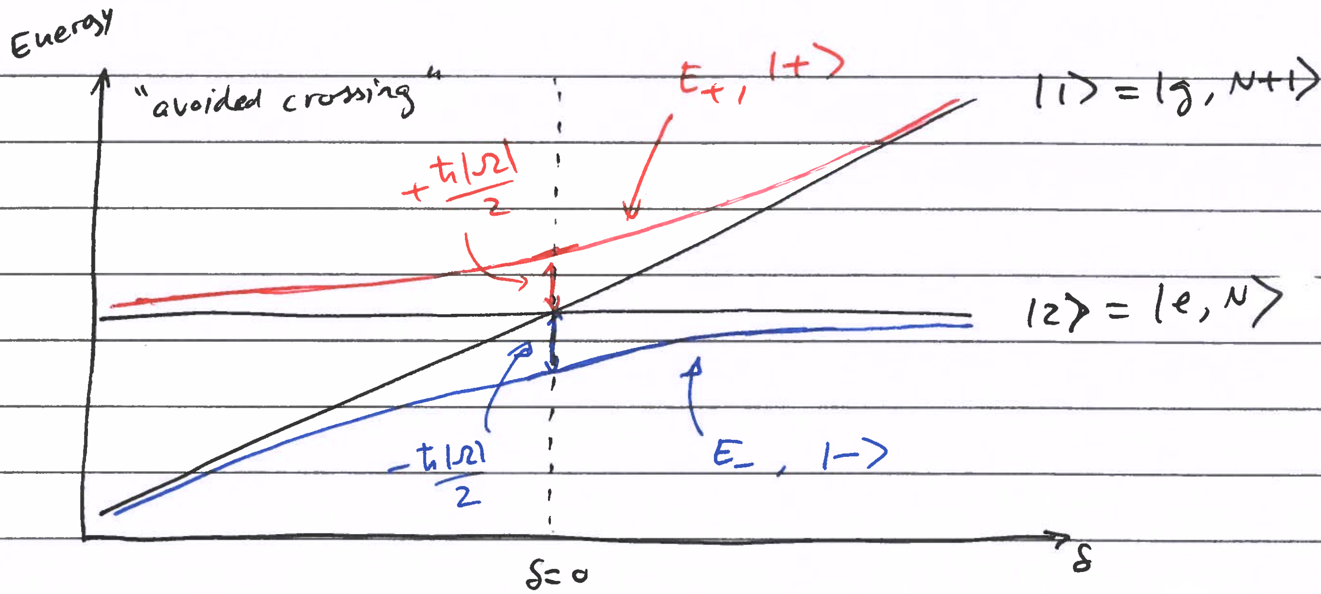
recall: $\sqrt{\delta^2 + |\Omega|^2} \simeq |\delta| + \frac{1}{2}\frac{|\Omega|^2}{|\delta|}$

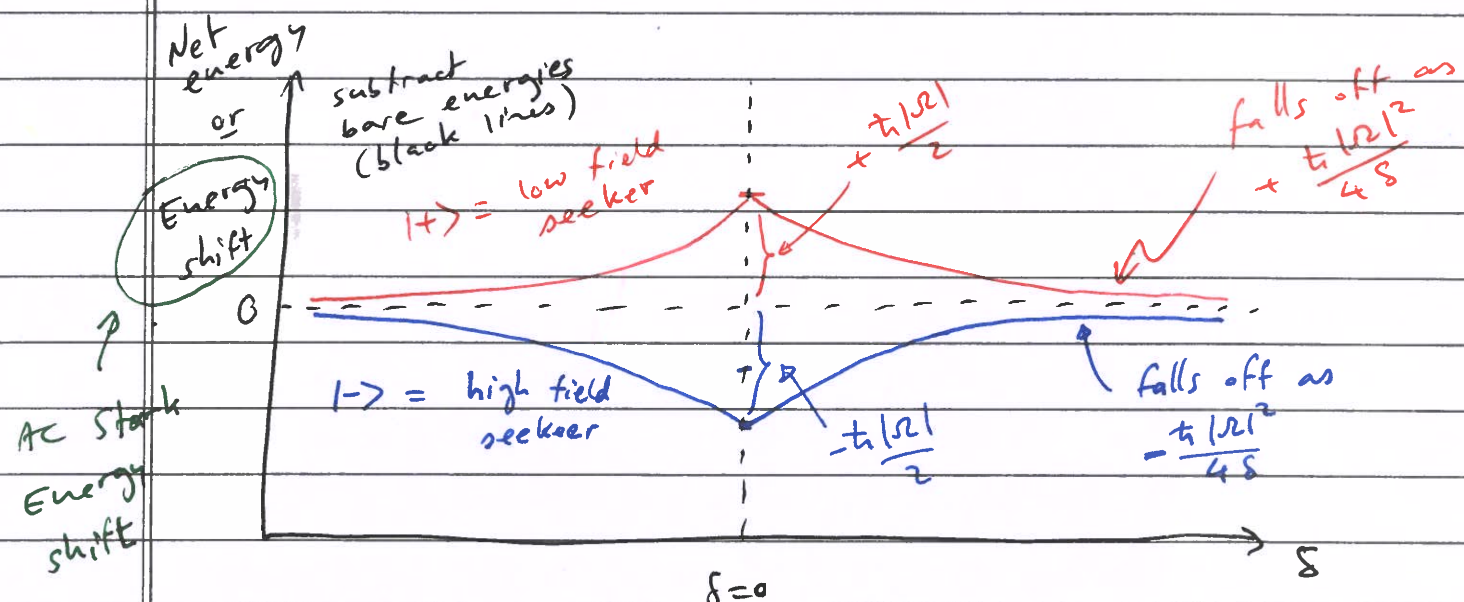
$$\Rightarrow E_\pm = \hbar \frac{\delta \pm \left(|\delta| + \frac{1}{2}\frac{|\Omega|^2}{\delta}\right)}{2}$$

$$\Rightarrow \quad E_+ = \hbar\left(\frac{\delta + |\delta|}{2}\right) + \frac{\hbar |\Omega|^2}{4\delta}$$

$$E_- = \hbar\left(\frac{\delta - |\delta|}{2}\right) - \frac{\hbar |\Omega|^2}{4\delta}$$

Laser/photon energy, does not depend on "Ω".





Next, let's find the eigenstates (or "dressed states")

$$\frac{E_+}{\hbar} = \lambda_+ = \frac{\delta + \Omega'}{2} \quad \text{where } \Omega' = \sqrt{\delta^2 + |\Omega|^2}$$

$$\begin{bmatrix} \delta - \lambda_+ & \Omega/2 \\ \Omega^*/2 & 0 - \lambda_+ \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$|g, N+1\rangle = |1\rangle$

$$\begin{bmatrix} \frac{\delta}{2} - \frac{\Omega'}{2} & \Omega/2 \\ \Omega^*/2 & -\frac{\delta}{2} - \frac{\Omega'}{2} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$|e, N\rangle = |2\rangle$

$$\Rightarrow \quad |+\rangle = \frac{\Omega}{2}|g, N+1\rangle - \left(\frac{\delta}{2} - \frac{\Omega'}{2}\right)|e, N\rangle$$

with normalization, we can write (for $\Omega \in \mathbb{R}$)

$$\boxed{\begin{aligned} |+\rangle &= \cos\theta\, |g, N+1\rangle + \sin\theta\, |e, N\rangle \\ |-\rangle &= -\sin\theta\, |g, N+1\rangle + \cos\theta\, |e, N\rangle \end{aligned}}$$

similarly,

with $\cos\theta = \dfrac{\Omega}{\sqrt{\Omega^2 + (\Omega' - \delta)^2}}$ and $\sin\theta = \dfrac{\Omega' - \delta}{\sqrt{\Omega^2 + (\Omega' - \delta)^2}}$

note: $\cos 2\theta = \cos^2\theta - \sin^2\theta = \dfrac{-\delta}{\Omega'}$

Example: On-resonance: $\delta = 0$, $\Omega' = \Omega$

$$\cos\theta = \frac{1}{\sqrt{2}}, \quad \sin\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \cos 2\theta = 0$$

$$\Rightarrow \quad \theta = \pi/4$$

$$\Rightarrow \begin{cases} |+\rangle = \dfrac{1}{\sqrt{2}}|g, N+1\rangle + \dfrac{1}{\sqrt{2}}|e, N\rangle \\[2ex] |-\rangle = -\dfrac{1}{\sqrt{2}}|g, N+1\rangle + \dfrac{1}{\sqrt{2}}|e, N\rangle \end{cases}$$

The $|+\rangle$ and $|-\rangle$ states are the eigenstates of the atom + photon system, but we perform measurements in the $\{|g, N+1\rangle, |e, N\rangle\}$ basis typically, so we observe oscillations at $\omega = \frac{\Delta E}{\hbar} = \Omega$, i.e. Rabi flopping.

The atom-light interaction mixes the $|g, N+1\rangle$ and $|e, N\rangle$ states and lifts their degeneracy.