Midterm on October 14 (Tuesday)

Thursday, October 2, 2025

Derivation of Spontaneous Emission Rate (continued)

[objective: show that $\frac{d}{dt} C_e = -\frac{\gamma}{2} c_e = -\frac{1}{2\tau} c_e$]

Last time we showed that the Schrodinger equation becomes:

$$\begin{cases} i \dfrac{d}{dt} C_e(t) = \displaystyle\sum_{\gamma \text{ states}} C_{g,\gamma}(t) \frac{\Omega_\gamma^*}{2} e^{-i\delta_\gamma t} & (1) \\ i \dfrac{d}{dt} C_{g,\gamma}(t) = \dfrac{\Omega_\gamma}{2} \cdot C_e(t) e^{i\delta_\gamma t} & (2) \end{cases}$$

infinite number of equations (1 for each γ photon state)

Next, we integrate (2):

$$C_{g,\gamma}(t) = -i \frac{\Omega_\gamma}{2} \int_0^t e^{i\delta_\gamma t'} C_e(t')\, dt' \qquad (3)$$

Then substitute (3) into (1):

$$\frac{d}{dt} C_e(t) = \underbrace{-\sum_{\gamma \text{ states}} \frac{|\Omega_\gamma|^2}{4} \int_0^t e^{-i\delta_\gamma (t-t')}}\, C_e(t')\, dt'$$

Plan: Evaluate this part in the hope of pulling $C_e(t')$ term out of integral

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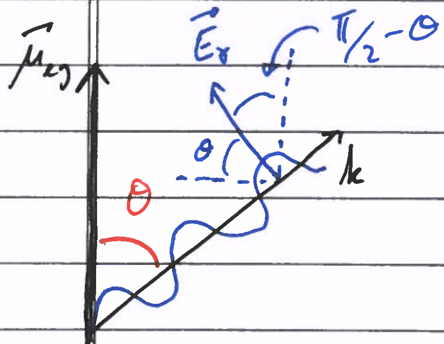
$\hat{z}_\gamma$, or $X\hat{x} \pm iY\hat{y}$

Recall: $\Omega_\gamma = \langle g | H_{int} | e \rangle = \dfrac{\langle g | -e\vec{R}\cdot\vec{E}_\gamma | e\rangle}{\hbar}$



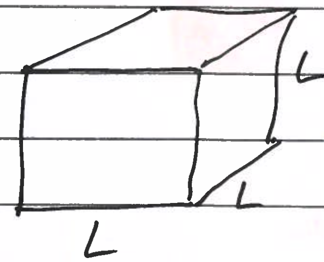
$$= \underbrace{-e\langle g|\vec{R}|e\rangle}_{\substack{\text{electric dipole}\\ \text{transition moment}\\ = \vec{\mu}_{eg}}} \cdot \frac{\vec{E}_\gamma}{\hbar} = -\vec{\mu}_{eg}\cdot\frac{\vec{E}_\gamma}{\hbar}$$

$$= -\frac{1}{\hbar}|\vec{\mu}_{eg}||\vec{E}_\gamma| \sin\theta$$

θ gives the direction of $\vec{k}$ not $\vec{E}$

Cartoon quantization of the EM-field

Universe = Box with side L =



$\Rightarrow$ volume = $V = L^3$

Energy of a photon in Box $= \hbar\omega_\gamma = \frac{1}{2}\int_V \left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) dv$

a little bit of algebra

$$\Rightarrow |\vec{E}_\gamma|_{RMS} = \sqrt{\frac{\hbar\omega_\gamma}{\varepsilon_0 V}}$$

So for Energy $= \frac{\hbar\omega_\gamma}{2} \Rightarrow |\vec{E}_{\gamma, N=0}|_{RMS} = |\vec{E}_{vacuum}|_{RMS}$

idea: vacuum has energy $\frac{\hbar\omega}{2}$

$$= \sqrt{\frac{\hbar\omega_\gamma}{2\varepsilon_0 V}}$$

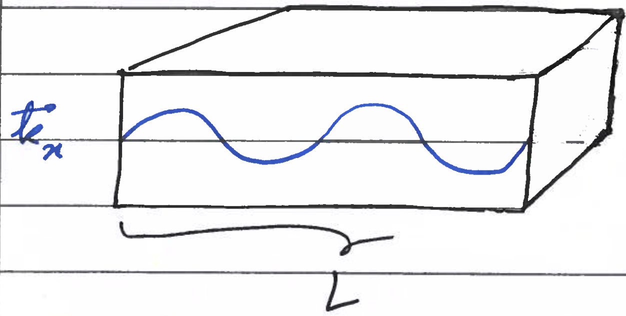
"vacuum" = $|0\rangle_\gamma$ photon state

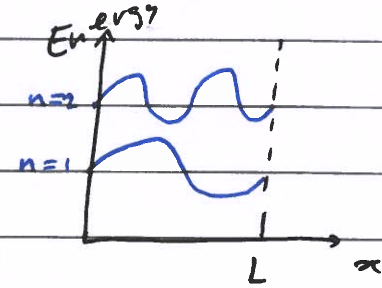
$$\Rightarrow |\vec{E}_{vacuum}| = \sqrt{2}\,|\vec{E}_{vacuum}| = \sqrt{\frac{\hbar\omega_\gamma}{\varepsilon_0 V}} \equiv |\vec{E}_\gamma|$$

Amplitude of the electric field of a photon

Converting "$\sum_\gamma$" to an integral (sum over γ states)

Let's count the number of photon spatial modes in the box universe with periodic boundary conditions





$$\Rightarrow \quad k_x = \frac{2\pi \, n_x}{L} \quad \Rightarrow \quad dn_x = \frac{L}{2\pi} dk_x$$

number of modes between k & $k+dk$

$$\Rightarrow \quad dN = \left(\frac{L}{2\pi}\right)^3 dk^3 = dn_x \, dn_y \, dn_z$$

2 polarizations

$$dN = 2 \frac{V}{(2\pi)^3} k^2 \sin\theta \, dk \, d\theta \, d\varphi$$

$k = \frac{\omega_\gamma}{c}$ $\qquad$ $\frac{d\omega_\gamma}{c}$

$$\Rightarrow \quad \underbrace{dN_\gamma = 2 \frac{V}{(2\pi)^3} \frac{\omega_\gamma^2}{c^3} \sin\theta \, d\omega_\gamma \, d\theta \, d\varphi}_{\text{Convert to integral}}$$

Thus: $$\frac{d}{dt} C_e(t) = -\sum_{\gamma \text{ states}} \frac{|\Omega_\gamma|^2}{4} \int_0^t e^{i\delta_\gamma (t-t')} C_e(t') \, dt'$$

recall: $\Omega_\gamma = -\frac{1}{\hbar} |\mu_{eg}| \underbrace{|\vec{E}_\gamma|}_{\sqrt{\frac{\hbar\omega_\gamma}{\varepsilon_0 V}}} \sin\theta \Rightarrow |\Omega_\gamma|^2 = \frac{|\mu_{eg}|^2}{\hbar} \frac{\omega_\gamma}{\varepsilon_0 V} \sin^2\theta$

$$\Rightarrow \frac{d}{dt} C_e(t) = -\int_0^\infty d\omega_\gamma \left\{ \frac{2V}{(2\pi)^3} \frac{\omega_\gamma^2}{c^3} \frac{|\mu_{eg}|^2}{4\hbar} \frac{\omega_\gamma}{\varepsilon_0 V} \underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \right.$$

$$\times \underbrace{\int_0^t e^{-i\delta_\gamma (t-t')}}_{} C_e(t')\,dt' \qquad \delta_\gamma = \omega_\gamma - \omega_{eg}$$

Strongly peaked at $\begin{cases} \omega_\gamma = \omega_{eg} \\ t = t' \end{cases}$

$$\to \pi\,\delta(\delta_\gamma)\,\tilde{\delta}(t-t') - \underbrace{P[i/\delta]}_{\text{Lamb shift (neglect)}}$$

$$\Rightarrow \frac{d}{dt} C_e(t) = -\frac{\cancel{2V}}{(2\pi)^{\cancel{3}}} \frac{\omega_{eg}^3}{c^3} \frac{\mu_{eg}^2}{\cancel{4}\hbar} \frac{1}{\varepsilon_0 \cancel{V}} \frac{\cancel{4}}{3}\, \cancel{2\pi}\, \cancel{\pi}\, C_e(t)$$

$$= -\frac{1}{2} \underbrace{\boxed{\frac{\mu_{eg}^2\, \omega_{eg}^3}{3\pi \varepsilon_0 \hbar c^3}}}_{\gamma = 1/\tau} C_e(t)$$

= spontaneous emission rate

$$\Rightarrow \boxed{\frac{d}{dt} C_e(t) = -\frac{1}{2}\gamma\, C_e(t)}$$

decay of C_e is exponential with time constant $= \frac{2}{\gamma}$

In "Schrodinger format", the equation becomes

$$i\hbar \frac{d}{dt} C_e(t) = -i\hbar \frac{\gamma}{2} C_e(t)$$

We would like to write the schrodinger equation of the dressed atom with spontaneous emission as

$$i\hbar \frac{d}{dt}\begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix} = \hbar \begin{pmatrix} \delta & \Omega/2 \\ \Omega^*/2 & 0 \end{pmatrix}\begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix} - \begin{pmatrix} 0 \; (?) \\ i\hbar \frac{\gamma}{2} c_e(t) \end{pmatrix}$$

We are not sure what to put here because a spontaneous emission depletes the excited state $|e\rangle$, atoms go into the ground state $|g\rangle$, but not coherently [the environment chooses a direction for the emitted photon]

"environment performs a measurement"

$$\Rightarrow i\hbar \frac{d}{dt}\begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix} = \underbrace{\hbar \begin{pmatrix} \delta & \Omega/2 \\ \Omega^*/2 & -i\gamma/2 \end{pmatrix}}_{H}\begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix}$$

imaginary term
↳ non-Hermitian

$\Rightarrow$ probability is not conserved!

⚠ The schrodinger equation can only handle coherent superpositions of states, not statistical mixtures.

Two methods for dealing with statistical mixtures:

- Density matrix
- Monte Carlo wavefunctions ("Quantum trajectories")

Density Matrix (von Neumann, Landau, Bloch)

definition: The density matrix ρ for a system with wavefunction $|\psi\rangle$ is given by

$$\boxed{\rho = |\psi\rangle\langle\psi|}$$

If $|\psi\rangle = c_g|g\rangle + c_e|e\rangle$, then

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} c_g c_g^* \, |g\rangle\langle g| & c_g c_e^* \, |g\rangle\langle e| \\ c_e c_g^* \, |e\rangle\langle g| & c_e c_e^* \, |e\rangle\langle e| \end{pmatrix}$$

For a population in a statistical mixture of states $|\psi_i\rangle$ with probabilities p_i, then

$$\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$

← allows for quantum superpositions and statistical mixtures simultaneously.

Stopped here

Main Properties:

Expectation value: $\langle A\rangle = \langle\psi|A|\psi\rangle = \mathrm{Tr}(\rho A)$

time evolution: $\boxed{i\hbar \frac{d}{dt}\rho = [H, \rho]}$

notation:

$$\rho = \begin{bmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{bmatrix} = \begin{bmatrix} c_g c_g^* & c_g c_e^* \\ c_e c_g^* & c_e c_e^* \end{bmatrix}$$