Tuesday, October 7, 2025

Density Matrix for modeling coherent & incoherent quantum dynamics

Definitions: $\rho = |\psi\rangle\langle\psi|$ for a wavefunction $|\psi\rangle$

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ for a statistical mixture of $|\psi_i\rangle$ states with probabilities p_i

Expectation value: $\langle A\rangle = \langle\psi|A|\psi\rangle = Tr(\rho A)$

Time evolution: $\boxed{i\hbar \frac{d}{dt}\rho = [H, \rho]}$

notation: $\rho = \begin{bmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{bmatrix} = \begin{bmatrix} c_g c_g^* & c_g c_e^* \\ c_e c_g^* & c_e c_e^* \end{bmatrix}$

Resonance Fluorescence — Optical Bloch Equations

The great advantage of the density matrix approach is that the decoherence that accompagnies spontaneous emission can be easily added in:

Coherent evolution: $i\hbar \frac{d}{dt}\rho = [H, \rho]$

with $H = \begin{bmatrix} E_g & \hbar\Omega\cos(\omega_L t) \\ \hbar\Omega^* \cos(\omega_L t) & E_e \end{bmatrix}$ + RWA + rotating frame

or $\underline{H = H_{dressed}} = \begin{bmatrix} \delta & \Omega/2 \\ \Omega^*/2 & 0 \end{bmatrix}$

Next, add in decoherence / spontaneous emission (decay rate γ)

$$\left\{ \begin{aligned} \frac{d}{dt}\rho_{ee} &= \frac{-i}{\hbar}[H,\rho]_{ee} - \gamma\rho_{ee} \\ \frac{d}{dt}\rho_{gg} &= \frac{-i}{\hbar}[H,\rho]_{gg} + \gamma\rho_{ee} \\ \frac{d}{dt}\rho_{eg} &= \frac{-i}{\hbar}[H,\rho]_{eg} - \frac{1}{2}\gamma\rho_{eg} \end{aligned} \right.$$

optical Bloch equations

$$\frac{d}{dt}\rho_{ge} = \frac{-i}{\hbar}[H,\rho]_{ge} - \frac{1}{2}\gamma_{ge}$$

these equations are equivalent

$\Rightarrow$ Solve for $\underset{P_{|g\rangle}}{\rho_{gg}}, \underset{P_{|e\rangle}}{\rho_{ee}}, \rho_{eg}$ $\rightarrow$ all continuous quantities

↳ wavefunction collapse is already included (due to decoherence)

↳ average output of many measurements

For generalization to multi-level atoms and multiple lasers see: Marquardt, Robinson, Hollberg JOSA B, 13, 1384 (1996).

We are interested in the <u>steady state</u> scattering rate:

$\rightarrow$ set $\frac{d}{dt}\rho_{ij} = 0 = \frac{-i}{\hbar}[H,\rho]_{ij} \pm \gamma \rho_{ij} \left(\frac{1}{2}\right)$

choose appropriately

$\Rightarrow$ solve for ρ_{ij}.

Scattering rate $= \gamma_{scattering} = \gamma P_e = \gamma \rho_{ee}$

probability to be in the excited state

$P_g = |c_g|^2 = \rho_{gg}$

$P_e = |c_e|^2 = \rho_{ee}$

some work (1 page-ish)

$$= \boxed{\frac{s_0}{1+s_0+\left(\frac{2\delta}{\gamma}\right)^2}\frac{\gamma}{2} = \gamma_{scattering}}$$

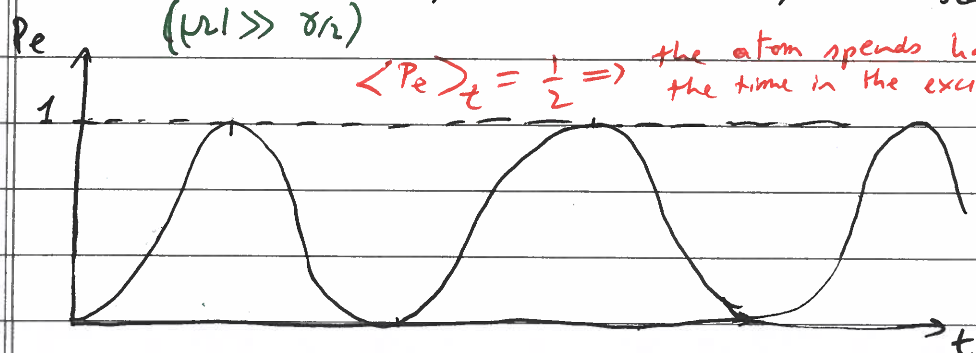
where $\begin{cases} s_0 = \text{saturation parameter} = \dfrac{I}{I_{sat}} = \dfrac{2|\Omega|^2}{\gamma^2} \\ I_{sat} = \dfrac{\pi h c}{3\lambda^2\tau} = 1.6\ mW/cm^2 \text{ for } ^{87}Rb \ (D_2 \text{ line}, 780\ nm) \end{cases}$

= intensity required to reach half of the maximum average upper state population in steady state (i.e. $\frac{1}{4}$)

<u>On Resonance</u> : $\delta = 0$

Case $I \gg I_{sat}$, i.e. $s_0 \to +\infty$, then $\gamma_{scattering} = \frac{\gamma}{2} = 19 \times 10^6$ photons/s (rubidium)

($|\Omega| \gg \gamma/2$)



$\langle P_e \rangle_t = \frac{1}{2} \Rightarrow$ the atom spends half the time in the excited state.

agrees well with experiment

(disagrees with classical prediction)

Case: $I \ll I_{sat}$, i.e. $s_0 \to 0$, then $\gamma_{scattering} \simeq s_0 \frac{\gamma}{2} \propto$ incident intensity
($|\Omega| \ll \gamma/2$)

good qualitative agreement with classical physics.

OFF - Resonance

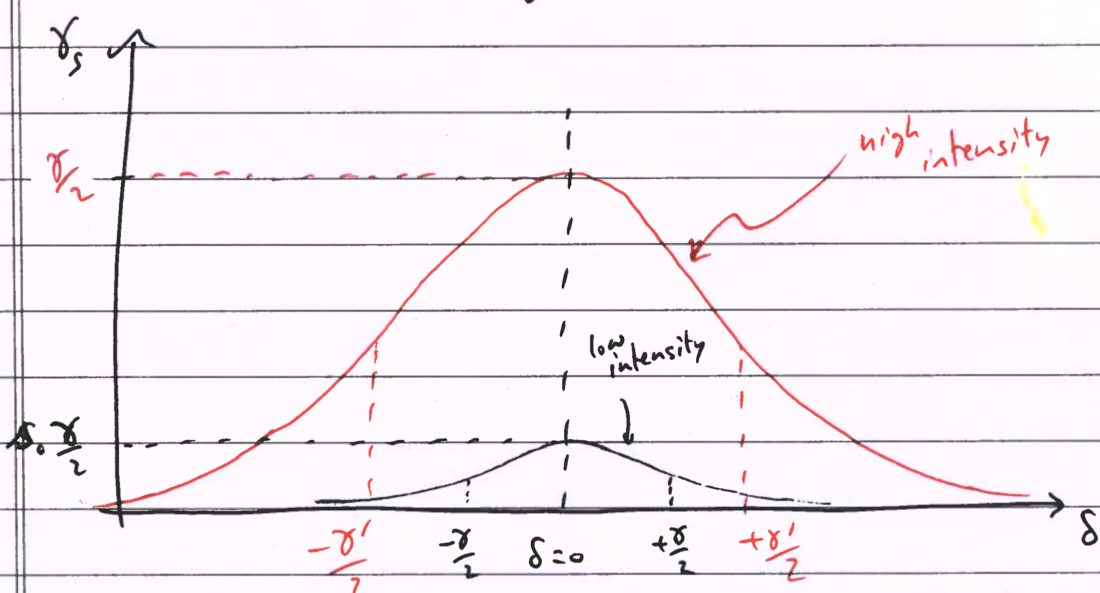
Linewidth for $s_0 \to 0$: $\gamma_s = \frac{1}{1+\left(\frac{2\delta}{\gamma}\right)^2} s_0 \frac{\gamma}{2}$
(low intensity limit)

$\Rightarrow$ Lorentzian with $FWHM = \gamma$

Linewidth for $s_0 \to +\infty$ "power broadening"
(high intensity limit)

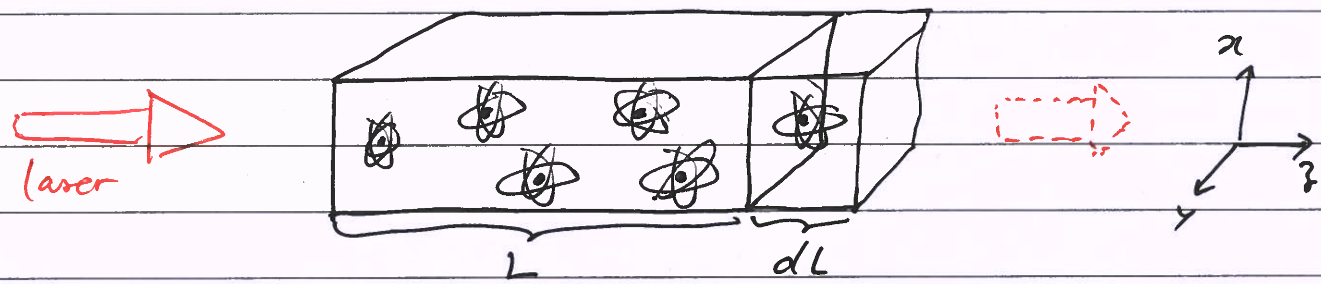
$$\gamma_s = \frac{s_0}{1+s_0} \frac{1}{1+\left(\frac{2\delta}{\gamma\sqrt{1+s_0}}\right)^2} \frac{\gamma}{2} \simeq \frac{1}{1+\left(\frac{2\delta}{\gamma'}\right)^2} \frac{\gamma}{2}$$

$\Rightarrow$ Lorentzian with $FWHM = \gamma' = \gamma\sqrt{1+s_0}$



Beer's law : Absorption in atomic vapor

Q: How is a laser attenuated as it travels through a gas of resonant atoms?



$$\text{gas density} = n = \frac{N_{atoms}}{\text{Volume}}$$

$$\text{Change in intensity} = dI = \frac{d\,\text{Power}}{\text{Area}} = -\frac{\hbar\omega_\ell \gamma_s \, n \cancel{A} \, dL}{\cancel{A}}$$

photons per atom

# of atoms in "dL" volume

Area of laser beam (cross-section)

$$\Rightarrow \quad \frac{dI}{dL} = -\hbar\omega_\ell \gamma_s n$$

In the weak intensity limit : $\gamma_s = \frac{I}{I_{sat}} \cdot \frac{\gamma}{2}$ (on resonance)

$$\Rightarrow \frac{dI}{dL} = -\hbar\omega_\ell \frac{\gamma}{2} \frac{n}{I_{sat}} \cdot I = -\frac{3\lambda^2}{2\pi} n I \quad \text{(on resonance)}$$

σ_{eg} = cross-section on resonance

$$= -\sigma_{eg} n I$$

$$\Rightarrow \quad \boxed{I(z) = I_0\, e^{-\sigma_{eg} n z}}$$ Beer's law

<u>note:</u> optical depth $= \sigma_{eg}\, n\, z$

$\hookrightarrow \quad I = I_0\, e^{-\text{optical depth}}$

<u>Application:</u> Absorption imaging of cold atoms.