

Tuesday, January 23, 2018

Jackson 6.1 - 6.3

Zangwill 15.3

Griffiths 10.1

Equations of Motion for $V \& \vec{A}$

$$\vec{\nabla}^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho(\vec{r}, t)}{\epsilon_0} \quad (1)$$

$$\left(\vec{\nabla}^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}(\vec{r}, t) \quad (2)$$

Recall $\vec{E} \& \vec{B}$ remain invariant under the gauge transformation

$$\begin{cases} \vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} \lambda \\ V \rightarrow V' = V - \frac{\partial \lambda}{\partial t} \end{cases}$$

where $\lambda(\vec{r}, t)$ is any well-behaved function

Coulomb gauge (also called the "transverse" gauge) proof:

we pick $\lambda(\vec{r}, t)$ such that

$$\begin{aligned} \vec{\nabla} \cdot \vec{A} &= 0 \\ \vec{\nabla} \cdot \vec{A}' &= \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{\nabla} \lambda = 0 \end{aligned}$$

$$\Rightarrow \vec{\nabla}^2 \lambda = -\vec{\nabla} \cdot \vec{A}$$

"poisson's equation"

In this case (1) $\rightarrow \vec{\nabla}^2 V + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -\frac{\rho(\vec{r}, t)}{\epsilon_0}$

$$\Rightarrow \vec{\nabla}^2 V = -\frac{\rho(\vec{r}, t)}{\epsilon_0} \Rightarrow V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t) d^3 r'}{|\vec{r} - \vec{r}'|}$$

\hookrightarrow note: $V(\vec{r}, t)$ depends instantaneously on $\rho(\vec{r}, t)$.

$$V(\vec{r}, t) \rightarrow 0 \text{ as } |\vec{r}| \rightarrow +\infty$$

Q: Are there speed of light issues?

A: No! \vec{A} does not share this ~~behavior~~ instantaneous

$$\begin{cases} \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

In fact, \vec{A} is still relatively hard to compute:

$$(2) \rightarrow \left(\vec{\nabla}^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}(\vec{r}, t)$$

$$\Rightarrow \vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial (\vec{\nabla} V)}{\partial t}$$

wave equation

known source term

can be calculated
from (1).

(sometimes referred to as "transverse current")

→ wave equation with source term → solve for \vec{A}
(see PHYS 611 next term)

note: if no sources are present ($J = 0, \vec{J} = 0$), then $V = 0$
and then \vec{A} satisfies:

$$\boxed{\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0}$$

(Standard wave equation)

→ Coulomb gauge is often used for quantizing the EM field
in free space, since you only need to quantize
 \vec{A} → often used in Quantum optics.

To see PHYS 622 later this semester (maybe)

Lorenz (or "Lorentz") Gauge → Lorentz Invariant " $\vec{A} \& V$ "

→ default gauge

we pick $A(\vec{r}, t)$ such that

$$\boxed{\vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}}$$

i.e. there is a 4-vector A^μ
for the electrodynamic potentials

proof that a $\lambda(\vec{r}, t)$ exists:

Given some $\vec{A} \notin V$, then find \vec{A}' , V' (i.e. λ) that satisfy the Lorentz gauge:

$$\vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t} = \vec{\nabla} \cdot (\vec{A} + \vec{\nabla} \lambda) = \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{\nabla} \lambda$$

~~plain~~
see problem set #1
on problem set #2

$$\Rightarrow \vec{\nabla}^2 \lambda = -\vec{\nabla} \cdot \vec{A} - \mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

$$\vec{\nabla} \cdot \vec{V} \cancel{\text{for } \lambda} \underbrace{\frac{\partial}{\partial t} (V - \frac{\partial \lambda}{\partial t})}_{\frac{\partial}{\partial t} V - \frac{\partial^2}{\partial t^2} \lambda}$$

$$\Rightarrow \vec{\nabla}^2 \lambda - \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2} = -\vec{\nabla} \cdot \vec{A} - \frac{1}{c^2} \frac{\partial V}{\partial t}$$

homogeneous

Solvable ... but not easy (wave equation with)
source term
see PHY5 611

$$(1) \rightarrow \vec{\nabla}^2 V + \underbrace{\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A})}_{-\mu_0 \epsilon_0 \frac{\partial V}{\partial t}} = -\frac{\rho(\vec{r}, t)}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{\nabla}^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho(\vec{r}, t)}{\epsilon_0}} \quad (i)$$

Also,

$$(2) \rightarrow \left(\vec{\nabla}^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J}(\vec{r}, t)$$

$$\Rightarrow \boxed{\vec{\nabla}^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}(\vec{r}, t)} \quad (ii)$$

Define the d'Alembertian: $\square^2 \equiv \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}$

$$= \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

thus the "Maxwell's equations" for \vec{A} & V become
(in the Lorenz gauge)

$$\begin{aligned} \square^2 V &= -\frac{\rho}{\epsilon} & (i) \\ \square^2 \vec{A} &= -\mu_0 \vec{J} & (ii) \end{aligned}$$

4 equations
and
4 unknowns

wave equations with ^{simple} source terms

A brief note on Retarded Potentials

Q: How do you solve (i') and (ii')?

In the static case $\square^2 = \nabla^2$ and we have $\begin{cases} \nabla^2 V = -\rho/\epsilon_0 & (i') \\ \nabla^2 \vec{A} = -\mu_0 \vec{J} & (ii') \end{cases}$

Solutions: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') d^3 r'}{|\vec{r} - \vec{r}'|}$ and $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') d^3 r'}{|\vec{r} - \vec{r}'|}$

time-dependent case

Ansatz (correct!): The potential at point \vec{r} is the ^{one} produced by $\rho(\vec{r}')$ and $\vec{J}(\vec{r}')$ at a retarded time $t_{r'} = t - \frac{|\vec{r} - \vec{r}'|}{c}$

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_{r'}) d^3 r'}{|\vec{r} - \vec{r}'|} \quad \text{and} \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_{r'}) d^3 r'}{|\vec{r} - \vec{r}'|}$$

This is only true in the Lorenz Gauge. Ansatz only works for potentials not fields (E & B).

Special Relativity and 4-vectors

(Jackson chpt. 11)

(Zangwill chpt 22)

(Griffiths chpt 12)

I- Einstein's Postulates for special relativity

1- The principle of relativity: The laws of physics are the same in all inertial reference frames.

inertial frame = coordinate system at constant velocity in a rest frame

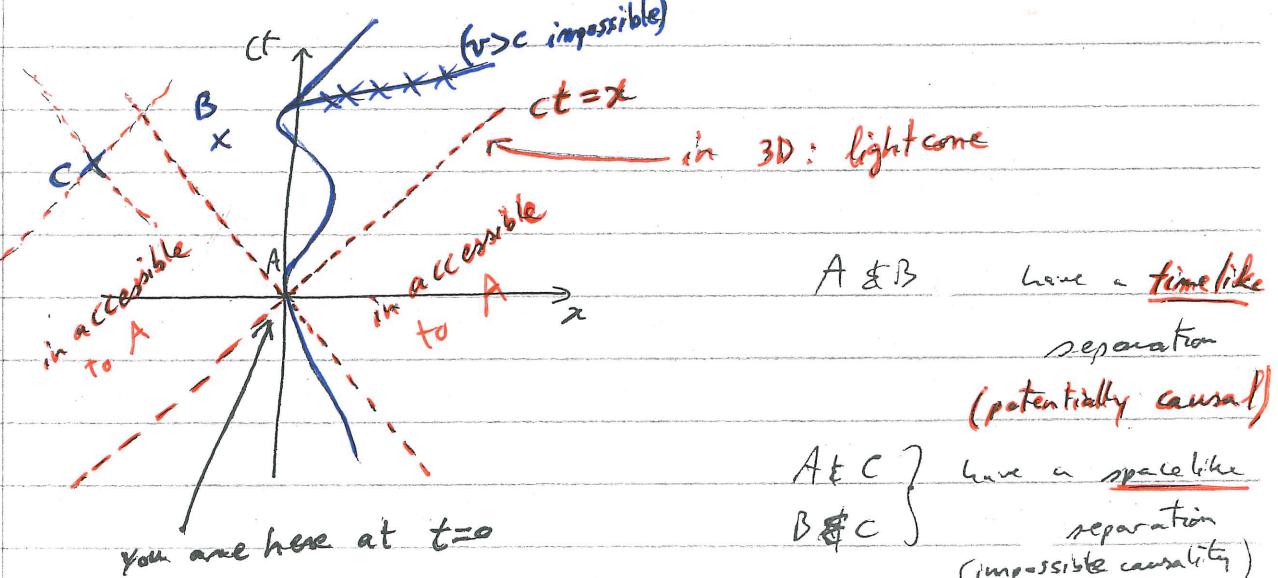
↳ a frame in which you cannot
(experimentally) determine if you are
moving based on local measurements
(i.e. if you are in box)

2- Universal Speed Light : The speed of light in vacuum is the same in all inertial frames, regardless of the motion of the source.

"4-space" "3+1 dimensions"

II Spacetime (i.e. Minkowski space)

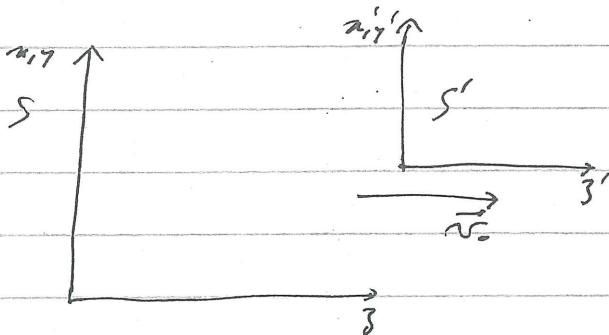
Consider a spatially 1D system ("1+" dimensions)



Note: the spacelike and timelike nature of 2 or more events ~~events~~ is unaffected by the reference frame.

III Lorentz Transformation of spacetime coordinates

Consider an inertial frame S' moving with velocity $\vec{v}_0 = v_0 \hat{z}$ in the S inertial frame. The origins overlap at time $t=0$ and $t'=0$.



The coordinates in the S and S' frames are related by

The Lorentz transformation: $(\gamma = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}, \beta = \frac{v_0}{c})$

$$\begin{array}{l} x' = x \\ y' = y \\ z' = \gamma(z - v_0 t) \Leftrightarrow z' = \gamma(1 - \frac{v_0}{c} ct) \\ t' = \gamma(t - \frac{v_0 z}{c^2}) \\ \Leftrightarrow ct' = \gamma(ct - \frac{v_0 z}{c}) \end{array} \quad \left| \begin{array}{l} (ct') \\ x' \\ y' \\ z' \end{array} \right\rangle = \left[\begin{array}{cccc} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{array} \right] \left| \begin{array}{l} (t) \\ x \\ y \\ z \end{array} \right\rangle$$

notes: $1 \leq \gamma \leq +\infty$, $0 \leq \beta \leq 1$

$\Delta(\vec{v}_0 = v_0 \hat{z})$ $\overset{\text{line}}{u}$ $\overset{\text{column}}{v}$

In 1+1 dimensions, we have $\begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ z \end{pmatrix}$

Rapidity

We define the rapidity γ (or boost parameter) as

$$\tanh(\gamma) = \beta$$

$$\text{recall } \sinh(\gamma) = \frac{e^\gamma - e^{-\gamma}}{2}$$

note: for $\{v_0 \rightarrow 0, \gamma \rightarrow 0\}$
 $\{v_0 \rightarrow c, \gamma \rightarrow +\infty\}$

$$\cosh(\gamma) = \frac{e^\gamma + e^{-\gamma}}{2}$$

note: $\cosh(\gamma)^2 - \sinh(\gamma)^2 = 1$

$$\tanh(\gamma) = \frac{\sinh(\gamma)}{\cosh(\gamma)} = \frac{e^\gamma - e^{-\gamma}}{e^\gamma + e^{-\gamma}}$$

one can show that $\gamma = \cosh(\eta)$ and $\gamma\beta = \sinh(\eta)$

$$\text{thus } \begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh \eta & -\sinh \eta \\ -\sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} ct \\ z \end{pmatrix}$$

The Lorentz transformation is reminiscent of a rotation, but with hyperbolic functions instead of trigonometric functions.

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2}} \\ &= \frac{1}{\sqrt{1-\frac{\sinh^2}{\cosh^2}}} = \frac{1}{\sqrt{\frac{\cosh^2 - \sinh^2}{\cosh^2}}} \\ &= \frac{\cosh}{\sqrt{\cosh^2 - \sinh^2}} = \cosh(\eta) \\ &\quad \boxed{\cosh^2 - \sinh^2 = 1} \end{aligned}$$

If we make the substitution $\eta = i\varphi$ also $\gamma\beta = \cosh \tanh = \sinh(\eta)$

$$\text{then } \begin{cases} \cosh \eta = \cos \varphi \\ \sinh \eta = i \sin \varphi \end{cases} \quad \begin{matrix} \uparrow \text{real} \\ \downarrow \text{imaginary} \end{matrix}$$

$$\text{thus } \begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \varphi & -i \sin \varphi \\ -i \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} ct \\ z \end{pmatrix}$$

we now move the "i" onto the "ct" coordinate

$$\begin{pmatrix} i'ct' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}}_{\text{rotation matrix}} \begin{pmatrix} i'ct \\ z \end{pmatrix}$$

\Rightarrow Lorentz transformation
can be thought of as a
rotation by an imaginary
angle φ ($\tanh(i\varphi) = \beta$)

see Goldstein chpt 7

on the modified spacetime

coordinate vector $\begin{pmatrix} i'ct \\ x \\ y \\ z \end{pmatrix}$

Lorentz transformation \equiv "rotation" in space time