

Thursday, February 8, 2018

Classical field theory: Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} - \partial^\beta \left( \frac{\partial \mathcal{L}}{\partial (\partial^\beta A^\alpha)} \right) = 0$$

gives the equations of motion for the field.

Lagrangian density

4-potential

Lagrangian density for EM fields

Zangwill 24.3  
Jackson 12.7

$$\mathcal{L}_{EM \text{ field}} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4\mu_0} 2(\vec{B}^2 - \frac{\vec{E}^2}{c^2}) \leftarrow \text{parity even}$$

Q: why not  $\mathcal{L}_{EM \text{ field}} = F_{\mu\nu} G^{\mu\nu} = -4 \frac{\vec{E} \cdot \vec{B}}{c}$  ?

A: - Gives zero in electrostatic & magnetostatic situations  
 - ~~this approach~~ is parity odd

$$\mathcal{L}_{interaction} = A_\mu J^\mu \leftarrow \text{charge-field interaction term}$$

$$\Rightarrow \mathcal{L}_{EM} = A_\mu J^\mu - \frac{1}{4\mu_0} [\partial_\mu A_\nu - \partial_\nu A_\mu] [\partial^\mu A^\nu - \partial^\nu A^\mu]$$

Q: Do we get Maxwell's equations from this formalism?

A: Yes... but 2 equations are automatic.

i.e. follow from definition of 4-potential

these are also given by  $\partial_\nu G^{\mu\nu} = 0$   
 source free Maxwell's eq.

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \stackrel{=0}{=} 0 \quad \text{no magnetic monopoles law}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \Rightarrow \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} V) - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} \stackrel{=0}{=} -\frac{\partial}{\partial t} \vec{B} \stackrel{= \vec{B}}{=} -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's law}$$

However Gauss's law & Ampère's law follow from the Euler-Lagrange eq.

$$\mathcal{L}_{EM} = J_\mu A^\mu - \frac{1}{4\mu_0} g_{\mu\gamma} g_{\nu\sigma} (\partial^\gamma A^\sigma - \partial^\sigma A^\gamma)(\partial^\mu A^\nu - \partial^\nu A^\mu) \quad \#2$$

(independent variables  
 $A^\mu, \partial^\nu A^\mu$ )

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = \frac{\partial}{\partial A^\alpha} J_\mu A^\mu = J_\mu \delta_\alpha^\mu = J_\alpha$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{4\mu_0} g_{\mu\gamma} g_{\nu\sigma} \left\{ (\delta_\beta^\gamma \delta_\alpha^\sigma F^{\mu\nu} - \delta_\beta^\sigma \delta_\alpha^\gamma F^{\mu\nu}) + (F^{\gamma\sigma} \delta_\beta^\mu \delta_\alpha^\nu - F^{\gamma\sigma} \delta_\beta^\nu \delta_\alpha^\mu) \right\}$$

product rule

$$= -\frac{1}{4\mu_0} \left\{ g_{\mu\beta} g_{\nu\alpha} F^{\mu\nu} - g_{\nu\alpha} g_{\mu\beta} F^{\mu\nu} + g_{\beta\gamma} g_{\alpha\sigma} F^{\gamma\sigma} - g_{\alpha\gamma} g_{\beta\sigma} F^{\gamma\sigma} \right\}$$

$$= -\frac{1}{4\mu_0} \left\{ \underbrace{F_{\beta\alpha}}_{-F_{\alpha\beta}} - F_{\alpha\beta} + \underbrace{F_{\beta\alpha}}_{-F_{\alpha\beta}} - F_{\alpha\beta} \right\}$$

$$= \frac{1}{4\mu_0} 4 F_{\alpha\beta} = \frac{1}{\mu_0} F_{\alpha\beta}$$

thus

$$\partial^\beta \left( \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} \right) = \frac{\partial}{\partial x_\beta} \left[ \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} \right] = \frac{1}{\mu_0} \partial^\beta F_{\alpha\beta}$$

consequently,

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = \partial^\beta \left( \frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} \right) \Rightarrow J_\alpha = \frac{1}{\mu_0} \partial^\beta F_{\alpha\beta}$$

$$\Rightarrow \boxed{\partial^\beta F_{\alpha\beta} = \mu_0 J_\alpha}$$

equivalent to Maxwell's eq.  
(see Feb. 1, 2018 lecture)

note:

$$\overbrace{\partial^\alpha (\partial^\beta F_{\alpha\beta})}^{=0} = \mu_0 \partial^\alpha J_\alpha$$

symmetric  $\partial^\alpha \partial^\beta = \partial^\beta \partial^\alpha$       ↓      anti-symmetric

$$\Rightarrow \partial^\alpha J_\alpha = 0$$

local conservation of charge  
(continuity equation)

Charge conservation and gauge invariance

$$\mathcal{L}_{EM} = J_\mu A^\mu - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

gauge change:  $\mathcal{L}'_{EM} = J_\mu A'^\mu - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$

$$= -\rho \frac{V''}{\epsilon} + \vec{J} \cdot \vec{A}' - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

$$= -\rho \left( V - \frac{\partial \lambda}{\partial t} \right) + \vec{J} \cdot (\vec{A} + \vec{\nabla} \lambda) - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

$\frac{1}{4\mu_0} 2(\vec{B} - \frac{\vec{E}^2}{c^2}) \leftarrow$  does not depend on gauge

$$= -\rho V + \vec{J} \cdot \vec{A} - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \rho \frac{\partial \lambda}{\partial t} + \vec{J} \cdot \vec{\nabla} \lambda$$

$$= J_\mu A^\mu - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \rho \frac{\partial \lambda}{\partial t} + \vec{J} \cdot \vec{\nabla} \lambda$$

$$= \mathcal{L}_{EM} + \underbrace{\rho \frac{\partial \lambda}{\partial t} + \vec{J} \cdot \vec{\nabla} \lambda}$$

$$S = \int d^4x \mathcal{L}_{EM}$$

not wrong { Does not contain  $A^\mu, \partial^\mu A^\nu$   
 so Euler-Lagrange eq. are unaffected  
 $\rightarrow$  too simplistic  $\rightarrow$  " $\lambda$ " represents a variation of  $A^\mu$ .

$$S' = \int d^4x \left[ \mathcal{L}_{EM} + \rho \frac{\partial \lambda}{\partial t} + \vec{J} \cdot \vec{\nabla} \lambda \right]$$

{

$A + \frac{\vec{\nabla} \lambda}{\epsilon \vec{A}}$

$V - \frac{\rho \partial \lambda}{\epsilon \partial t}$

$S_V$

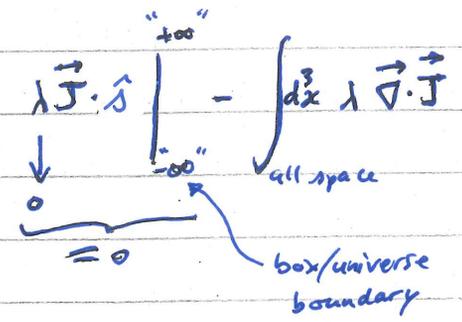
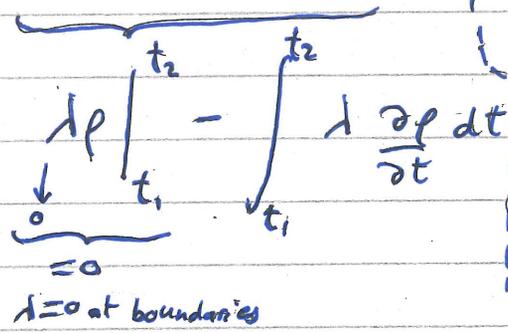
$$\delta S = 0 = S - S' = - \int d^4x \left[ \rho \frac{\partial \lambda}{\partial t} + \vec{J} \cdot \vec{\nabla} \lambda \right]$$

$$\delta S = 0 = - \int d^3x \int_{t_1}^{t_2} dt \rho \frac{\partial \lambda}{\partial t} - \int (cdt) \int d^3x \overbrace{\vec{J} \cdot \vec{\nabla} \lambda}^{J_x \partial_x \lambda + J_y \partial_y \lambda + J_z \partial_z \lambda}$$

IBP:  $u = \rho, dv = \frac{\partial \lambda}{\partial t} dt$

$u = \vec{J}, dv = \vec{\nabla} \lambda d^3x$   
 $du = \frac{\partial \rho}{\partial t} dt, v = \lambda$

$du = \frac{\partial \rho}{\partial t} dt, v = \lambda$



$$\delta S = 0 = \int d^4x \left\{ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right\} \lambda$$

$\partial_\mu J^\mu$

i.e. any gauge function with  $\lambda = 0$  at boundary of space-time

↳ this must be true for any  $\lambda$  ( $\lambda \equiv \delta A^\mu$ )  
 so the integrand  $\{ \}$  must be zero

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \Rightarrow \text{charge is locally conserved!}$$

gauge invariance (of  $\delta S$ )  $\iff$  local conservation of charge

continuous symmetry  $\rightarrow$  conserved "charge"/"current"  
 conserved quantity

↳ example of Noether's theorem

[We take a short break from  $E \notin M$ ]

disciple of Hilbert

Noether's Theorem (Emmy Noether, 1915, 1918.)

principle / method

[see Peskin & Schroeder and Goldstein]

Consider the continuous transformation of a field  $\phi(x)$   
(deformation/perturbation)

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

Scalar field

potentially infinitesimal parameter

a deformation of the field  $\phi(x)$

this transformation is called a continuous symmetry of the field/system if it does not change the field equations of motion.

simple example

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2$$

$$\phi(x) \rightarrow \phi' = \phi(x) + \eta, \quad \eta = cst$$
  
$$\alpha \Delta \phi(x)$$

$$\mathcal{L}' = \frac{1}{2} [\partial_\mu (\phi + \eta)]^2 = \frac{1}{2} (\partial_\mu \phi)^2 = \mathcal{L} \Rightarrow \text{clearly, equations of motion are unaffected}$$

The field equations of motion remain invariant if ~~the transformed Lagrangian~~ action remains unchanged, i.e. if the transformed Lagrangian becomes

$$\mathcal{L}(\phi(x)) \rightarrow \mathcal{L}(\phi'(x)) = \mathcal{L}(\phi(x)) + \alpha \Delta \mathcal{L}'$$

where

$$\begin{cases} \Delta \mathcal{L}' = 0 \\ \text{or} \\ \Delta \mathcal{L}' = \partial_\mu J^\mu = \frac{\partial}{\partial x^\mu} J^\mu \end{cases}$$

some general "current"

you can generally figure this out by direct calculation

Stopped here